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RADIATIVE CORRECTIONS TO $K \to \mu + \pi + \nu$ WITH FINITE NEUTRETTO MASS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled Radiative Corrections to $K \to \mu + \pi + \nu$ with Finite Neutretto Mass, submitted by Nan Nyoh Wong, in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

All the radiative corrections are calculated to second order in elementary charge e for the decay $K^{\pm} \to \mu^{\pm} + \pi^{0} + \nu$ with finite neutretto mass. The infrared divergent terms arising from the virtual photon processes are only partially cancelled by that from the real photon emission process. This is due to the phase space restrictions imposed on real photon emission. Therefore, the corrections would contain a logarithmic term which diverges at the high energy end of the pion or muon spectrum. This term is expected to make a significant contribution to the spectrum of pion and muon, if the charged lepton in the final state is relativistic. The ratios of the spectrum with $m_{\nu} \neq 0$ to that with $m_{\nu}=0$ for both muon and pion, and the ratio of energy correlations between them are calculated. A rough estimate of the effect of the corrections due to the divergent terms on the corresponding ratios are given so as to illustrate the effect of the correction to the phase space ratio for determining the mass of the neutretto.

ACKNOWLEDGEMENTS

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CONTENTS

NOTATION

The following notation will be used throughout this thesis.

1. The matric $g_{\mu\nu}$ has the following components.

$$g_{oo} = 1 \qquad g_{kk} = -1 \qquad \text{for} \quad k = 1, 2, 3.$$

$$g_{\mu\nu} = 0 \qquad \text{for} \quad \mu \neq \nu$$

2. The relativistic notations are as follows:

$$(\gamma_o)^2 = -(\gamma_k)^2 = 1 \qquad k = 1, 2, 3.$$

$$\gamma_5 = \gamma_o\gamma_1\gamma_2\gamma_3 \qquad (\gamma_5)^2 = -1$$

3. Natural units have been used i.e. $\hbar = c = 1$.

4. The notation for the 4-momentum, 3-momentum, energy and mass of the particles is as follows:

	4-momentum	3-momentum	energy	mass
K meson	K	0	m_K	m_K
muon	p	$\bar{p}$	E	m_μ
pion	k	$\bar{k}$	ω	m_π
neutretto	q	$\bar{q}$	ϵ	m_γ
real photon	k'	$\bar{k}'$	ω'	-
virtual photon	t	$\bar{t}$	t_o	-

5. U is the spinor of muon.

   V is the spinor of neutretto.

6. G' is the coupling constant.

7. G is the 4-momentum imparted to pion and neutretto, $G = K - p = k + q$.

8. Λ is the cut off mass used in Feynman regulator.

9. λ is the fictitious photon mass, which is introduced to avert the infra-red divergence.

10. ε is the polarization vector of the real photon.

11. $\alpha = e^2/4\pi$ is the fine structure constant = 1/137.

12. θ is the angle between $\bar{p}$ and $\bar{k}'$.

13. θ' is the angle between $\bar{q}$ and $\bar{k}'$.

14. φ is the angle between $\bar{p}$ and $\bar{q}$.

15. ξ is the angle between $\bar{k}$ and $\bar{k}'$.

16. η is the angle between $\bar{p}$ and $\bar{k}$.

CHAPTER I INTRODUCTION

An upper limit of about 3 Mev has so far been placed on the neutretto (muon neutrino) mass[1]. It is worthwhile to study, theoretically, a process that would give a much lower bound on the neutretto mass and would at the same time be experimentally feasible.

It has been suggested by several authors[2,3,4,5] that the upper limit on the neutretto mass can best be obtained from an observation of the shape of the spectrum of one of the final state particles in the three-particle decays of the type

$$A \longrightarrow B + C + \nu$$

near the high energy end.

In general, the spectrum of the particle $dP(E,m_\nu)$ of one of the final state particles, B or C with finite neutretto mass $m_\nu \neq 0$, would be different from the spectrum $dP(E,0)$ with $m_\nu = 0$, for there is less phase space available than if $m_\nu \neq 0$.

Let us calculate the phase space integration in the leptonic decay $K^{\pm} \longrightarrow \pi^{o} + \mu^{\pm} + \nu$

$$\int \frac{d^3p}{2E}\,\frac{d^3k}{2\omega}\,\frac{d^3q}{2\epsilon}\,\delta^4(K-p-k-q) \qquad (1)$$

Where K, p, k and q are the four momenta of K meson, muon, pion and neutretto respectively, and E, ω and ϵ are the energy of muon, pion and neutretto respectively. If we do not carry out the p integration then we would get the muon spectrum $dP(E,m_\nu)$

$$dP(E,m_\nu) = \int \frac{d^3k}{2\omega} \frac{d^3q}{2\epsilon} \delta^4(K-p-k-q) \tag{2}$$

To work out the integral, we set G=K-p (i.e. the four momentum imparted to the neutretto and pion during the decay) and work in the time-like-G-frame defined by $G=(G_o,0,0,0)$ (Appendix A), we have

$$dP(E,m_\nu) = \frac{\pi}{2G}\left[G^4 - 2G^2(m_\pi^2 + m_\nu^2) + (m_\pi^2 - m_\nu^2)^2\right]^{\frac{1}{2}} \tag{3}$$

In the case where $m_\nu=0$, if one evaluates the integral in the frame where K meson decays at rest, i.e. $K=(m_K,0,0,0)$, then

$$G^2 = (K - p)^2 = m_K^2 + m_\mu^2 - 2m_KE \tag{4}$$

Since we are interested in the behaviour of muon energy spectrum near the end-point, G^2 should be close to its minimum value. This follows from eq. (4) where G^2 is minimum when E is maximum. The minimum value of G^2 can be evaluated from

$$G^2 = (K - p)^2 = (q + k)^2 \tag{5}$$

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The right hand side of eq. (5) has a minimum value m_π^2 for $m_\nu=0$. If E is slightly less than the maximum value, say $E = E_{max} - \delta E$, then G^2 will be slightly above G^2_{min}, say $G^2 = G^2_{min} + \Delta$; therefore dP(E,0) close to the E_{max} will behave like

$$dP(E,0) = \frac{\pi}{2m_\pi^2}\left[(m_\pi^2+\Delta)^2 - 2(m_\pi^2+\Delta)m_\pi^2 + m_\pi^4\right]^{\frac{1}{2}}$$

$$\approx \frac{\pi}{2m_\pi^2}(\Delta^2)^{\frac{1}{2}} \tag{6}$$

i.e. proportional to the first power of Δ.

If we had not set $m_\nu=0$, then

$$dP(E,m_\nu) \approx \frac{\pi}{2m_\pi^2}\left[(m_\pi^2+\Delta)^2 - 2(m_\pi^2+\Delta)(m_\pi^2+m_\nu^2) + (m_\pi^2-m_\nu^2)^2\right]^{\frac{1}{2}}$$

$$\approx \frac{\pi}{2m_\pi^2}(\Delta^2 - 4m_\pi^2 m_\nu^2)^{\frac{1}{2}} \tag{7}$$

In the above expression terms of higher order in m_ν have been neglected. From eq. (6) and eq. (7) we can get the phase space ratio

$$\frac{dP(E,m_\nu)}{dP(E,0)} \approx \left[1 - \frac{4m_\pi^2 m_\nu^2}{\Delta^2}\right]^{\frac{1}{2}} \tag{8}$$

This can also be expressed in terms of δE^2 instead of Δ^2. From eq. (4)

$$E = \frac{m_K^2 + m_\mu^2 - G^2}{2m_K} \tag{9}$$

Now, when $G^2 = G^2_{min} + \Delta$, then

$$E = \frac{m_K^2 - m_\pi^2 + m_\mu^2}{2m_K} - \frac{\Delta}{2m_K} = E_{max} - \delta E \tag{10}$$

therefore

$$\delta E = \Delta/2m_K \tag{11}$$

and

$$\frac{dP(E,m_\nu)}{dP(E,0)} = \left[1 - \frac{m_\pi^2 m_\nu^2}{m_K^2(\delta E)^2}\right]^{\frac{1}{2}} \tag{12}$$

This expression is quite general for all the three-particle decays. From eq. (12) we see that the muon spectrum cuts out at a distance $\delta E = m_\pi m_\nu/m_K$ from the end point for the zero mass case. The linear dependence of δE on m_ν is an experimental advantage. On the other hand, the $dP(E,m_\nu)$ differs from $dP(E,0)$ also due to the dependence of the matrix elements on m_ν, if $m \neq 0$. But if it is assumed that the leptons couple locally and the leptonic current is of the type $\bar{U}\gamma_\mu(1-i\gamma_5)V$, then the contribution to the spectrum arising from the m_ν-dependent term in the matrix element is always quadratic in m_ν while the contribution arising from the phase space considerations alone is linear in m_ν. Thus for small m_ν the phase space contribution dominates the contribution coming from the matrix element.

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If one plots $dP(E,m_\nu)/dP(E,0)$ versus energy of the observed particle, and looks for the deviations of the spectrum at the end-point from the horizontal straight line appropriate to $m_\nu=0$, a limit on m_ν can be set.

Six different spectra in the decays $\Lambda^0 \to p + \mu^- + \nu$, $\pi^\pm \to \mu^\pm + \nu + \gamma$, $K^\pm \to \pi^0 + \mu^\pm + \nu$ were suggested in both the papers by Dennery and Primakoff (1963)(2) and Ginsberg (1965)(5). Greatest deviations may be anticipated in the spectra from the first two decays, but because of the relatively short lifetime and electrical neutrality of the Λ^0 and because of the difficulty of making precise measurement of γ energy, the spectra from $K^\pm \to \pi^0 + \mu^\pm + \nu$ would be more favorable to study experimentally.

To see the effect on the spectrum of radiative correction on the decay, let us take the case where there is a real photon being emitted with four momentum k' (innerbremsstrahlung). If E is close to E_{max}, that is to say we only look at the end-point energy in the spectrum at which the effect on m_ν is significant, then the photon must not be allowed to carry away a lot of energy, and must therefore be a soft photon. The phase space integral then becomes

$$\int \frac{d^3p}{2E}\,\frac{d^3k}{2\omega}\,\frac{d^3q}{2\epsilon}\,\frac{d^3k'}{2\omega'}\,\delta^4(K-k'-p-k-q) \qquad (13)$$

Following the same procedure as used before, with the replacement of G by $G' = G-k'$, we have

$$dP(E,m_\nu) = \frac{\pi}{2G^2}\int \frac{d^3k'}{2\omega'}\left[(G'^4 - 2G'^2(m_\pi^2+m_\nu^2) + (m_\pi^2-m_\nu^2)^2\right]^{\frac{1}{2}} \tag{14}$$

To illustrate the importance of the radiative corrections. It is sufficient to deal with the case $m_\nu=0$. One has by energy-momentum conservation

$$(G - k')^2 = (q + k)^2 \tag{15}$$

In the time-like G-frame $G = (G_o,0,0,0)$, eq. (15) becomes

$$G_o^2 - 2k_o'G_o = (q + k)^2 \tag{16}$$

(k_o' is maximum when $(q+k)^2$ is a minimum or G^2 is a minimum.) If we allow G^2 to be $m_\pi^2 + \Delta$ as done earlier the maximum of k_o' is determined by eq. (16) as

$$m_\pi^2 + \Delta - 2k_o'\Big|_m G_o = m_\pi^2$$

or

$$k_o'\Big|_{max} = \Delta/2m_\pi \tag{17}$$

$$dP(E,0) \sim \int_o^{\Delta/2m_\pi} k_o'\left[(G-k')^4 - 2(G-k')^2m_\pi^2 + m_\pi^4\right]^{\frac{1}{2}} dk_o' \tag{18}$$

by expanding the terms in the bracket, and neglecting terms of order $k_o'^2\Delta^2$ in favour of terms of order Δ^2 or $k_o'^2$, we get

$$dP(E,0) \sim \int_0^{\frac{\Delta}{2m_\pi}} k_0' \left[\Delta^2 + \Delta G_0 k_0' - 4\Delta G_0 k_0'\right] dk_0'$$

$$\sim \int_0^{\frac{\Delta}{2m_\pi}} k_0' \text{ (Term of order } \Delta) \; dk_0'$$

$$\sim \Delta^3$$

Thus in this case $dP(E,0)$ is proportional to the third power of Δ. This result is quite different from the case where the real photon is absent. Because of the lack of phase space available to the real photon, the emission of real photon is inhibited at the end point. Therefore the infrared divergent terms arising from the radiative corrections of the virtual photon processes will be only partially cancelled by those from the real photon emission. Thus the corrections would reflect this lack of cancellation of the divergent terms through an infinity at the end point of the spectrum. It has been shown (Chapter V) that such a term does indeed appear in the form $\frac{e^2}{4\pi} B(\beta)\log(1- \frac{m_\pi m_\nu}{m_K \delta E})$, where $B(\beta) = 1 + \frac{1}{2\beta}\log\frac{1-\beta}{1+\beta}$ with $\beta = \bar{p}/E$ which is finite for the range of momenta available to the muon, while $\log(1 - \frac{m_\pi m_\nu}{m_K \delta E})$ diverges logarithmically at the end point of the muon spectrum. This effect was first anticipated by Allcock (1965)[3] in his work on radiative corrections to the decay $\mu \longrightarrow e + \nu + \tilde{\nu}$. Later, Kamal (1965)[4] worked on the decay $\pi \longrightarrow \mu + \nu + \gamma$ and showed that this correction term will not have an appreciable effect on the ratio $dP(E,m_\nu)/dP(E,0)$. This happens because

in the $\pi_{\mu 3}$ case, the charged lepton is not relativistic, or in other words, the radiative corrections are damped out by the small value of function $B(\beta_m) = 1 + \frac{1}{2\beta_m} \log \frac{1-\beta_m}{1+\beta_m}$. He also suggested in the paper that the radiative corrections would be expected to be large in the case of $K_{\mu 3}$ decay, for the mass of K meson is large, and therefore the charged lepton can become relativistic (i.e. $\beta_m \approx 1$). Hence $B(\beta_m)$ would be of order unity. Consequently the logarithmically divergent term will not be damped out and would be important.

In Chapter II a brief account is given of the derivation of a model interaction Hamiltonian. The model Hamiltonian is justified through the use of the intermediate vector boson hypothesis. With this assumption some simplifications are made for the convenience of later calculations. Our results are expected to be independent of the form of the Hamiltonian.

All virtual photon corrections are made in Chapter III and real photon corrections in Chapter IV.

The ratios of the spectrum with $m_\nu \neq 0$ to that with $m_\nu = 0$ for both the muon and pion are calculated in Chapter V. The energy correlation between the pion and muon is also dealt with in this chapter.

CHAPTER II FORM OF THE AMPLITUDE

The decay mode of $K^{\pm} \rightarrow \mu^{\pm} + \pi^{0} + \frac{\nu}{\bar{\nu}}$ can be described by a Feynman amplitude whose general form is restricted by Lorentz Invariance and Time-reversal Invariance. We shall assign a local leptonic current of the (V-A) type, $\bar{U}_{\ell}\gamma_{\alpha}(1-i\gamma_5)V_{\nu}$, to the lepton pair. The decay process then looks like fig. 1.

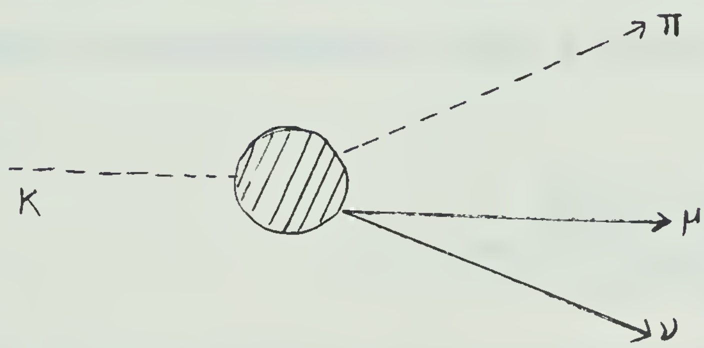

Fig. 1

The strong interaction involved in the transition to the intermediate state with the emission of the pion is not known. But we can write down phenomenologically a current-current type interaction. The decay matrix is then

$$\langle K|gJ_{\alpha}|\pi^{0}\rangle\bar{U}_{\mu}\gamma_{\alpha}(1-i\gamma_5)V = g\left[c(K+k)_{\alpha}+d(K-k)_{\alpha}\right]\bar{U}\gamma_{\alpha}(1-i\gamma_5)V \qquad (19)$$

where c and d are functions of $G = K-k$, and are real if time reversal invariance holds for the decay interaction; g is the coupling constant.

We can make some simplifications to the Feynman amplitude by

the assumption that an intermediate vector boson state exists. Then, the diagram would be

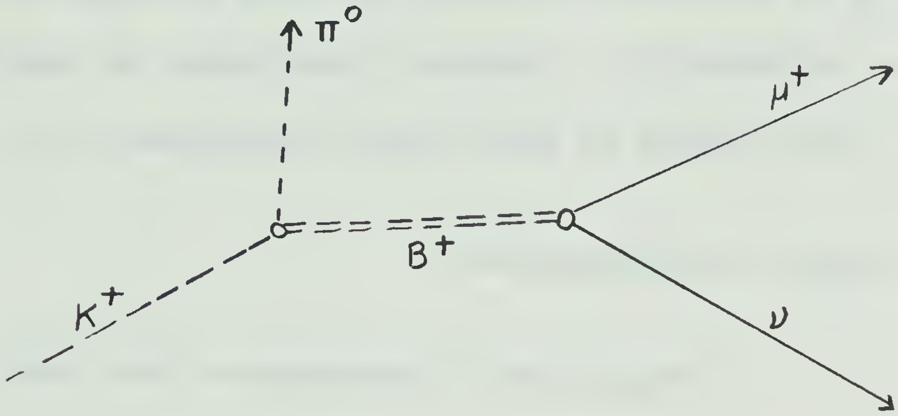

Fig. 2

Assuming a phenomenological coupling between the boson B and the (πK) system, the contribution of Fig. 2 to the decay matrix would be of the form

$$(K - k)_\alpha \frac{\overline{U}\, \gamma_\alpha(1 - i\gamma_5)V}{(K - k)^2 - m_B^2} \quad .$$

If $m_B^2 \gg (K - k)^2$ then the denominator is essentially a constant and the decay matrix can be simulated by a phenomenological weak interaction[6]

$$G'(\phi_\pi \partial_\alpha \phi_K - \phi_K \partial_\alpha \phi_\pi)\overline{U}\, \gamma_\mu (1 - i\gamma_5)V \quad . \tag{20}$$

where G' is the coupling constant. Note that equation (20) is analogous to equation (19) if the constant d in equation (19) is zero.

The electromagnetic interaction is introduced through the replacement of ∂_μ by $\partial_\mu + ieA_\mu$ in equation (20). Hence, we have

$$G'\left[\phi_\pi(\partial_\mu + ieA_\mu)\phi_K - \phi_K\partial_\mu\phi_\pi\right]\bar{U}\gamma_\mu(1 - i\gamma_5)V \qquad (21)$$

In equation (21) the operator operating on ϕ_π is unchanged, because the pion is electrically neutral. By expanding eq. (21) we see that there is an additional term, which is absent in eq. (20).

$$ieG'\phi_\pi A_\mu\phi_K\bar{U}\gamma_\mu(1 - i\gamma_5)V \qquad (22)$$

This term corresponds to the diagram

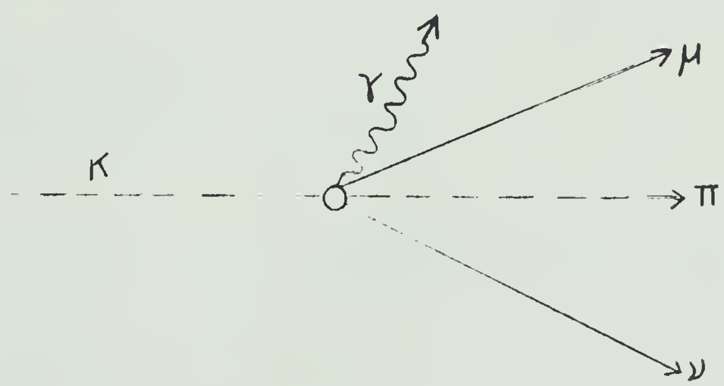

Fig. 3

As a consequence of this result, the diagrams of the virtual photon processes with the photon emitted at the vertex, like fig. 5 and fig. 7, should be evaluated and included in the corrections.

It is expected that the results we have on the ratio $dP(E,m_\nu)/dP(E,0)$ would be independent of the model. This expectation can be made plausible on the grounds that had we chosen the more general matrix element of eq. (19) the important radiative corrections would appear as a multiplicative factor to the entire matrix element, leaving the ratio $dP(E,m_\nu)/dP(E,0)$ unaltered. The important radiative correc-

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tions come from the innerbremsstrahlung graph where the photon is emitted from the charged lepton leg (see Chapter IV).

CHAPTER III VIRTUAL PHOTON CORRECTIONS

The perturbation theory will be used in the calculation of the radiative corrections of the $K^{\pm} \longrightarrow \mu^{\pm} + \pi^{o} + \nu$ decay. We are interested only in the radiative corrections of order e^2, because contributions from the higher order terms are small enough to be neglected (Allcock (1965))[3].

3.1 THE SELF-ENERGY OF K MESON

First, let us consider the self-energy of the K meson. The Feynman diagrams corresponding to this process are fig. 4 and fig. 5. The reason for inclusion of fig. 5 has been given in Chapter II.

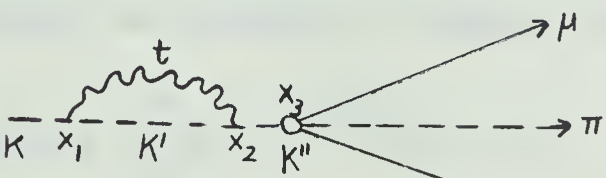

Fig. 4

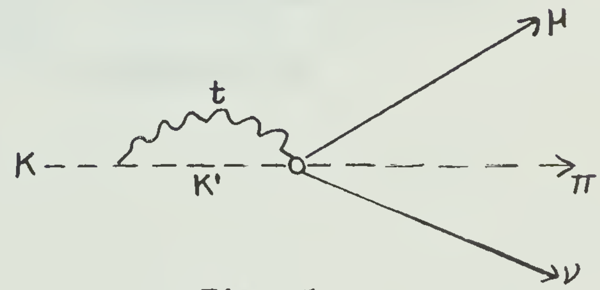

Fig. 5

The matrix element corresponding to fig. 4 is:

$$\int -iG'\bar{U}\gamma_{\mu}(1 - i\gamma_5)(K'' + k)^{\mu} \frac{1}{(2\pi)^4} \frac{1}{(K''^2 - m_K^2)} (-ie)(K'' + K')^{\rho} \frac{1}{(2\pi)^4} \frac{1}{K'^2 - m_K^2}$$

$$\frac{-i}{(2\pi)^4}\frac{1}{t^2-\lambda^2}(-ie)(K'+K)_\rho V(2\pi)^4\delta(K''-p-k-q)(2\pi)^4\delta(K'+t-K)(2\pi)^4$$

$$\delta(K-t-K')d^4K''d^4K'd^4t. \tag{23}$$

In the above expression we have omitted the normalization factors such as $\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m_i}{E_i}}$ for the fermions and $\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_j}}$ for the bosons. As we are going to take the ratio of the spectrum, these factors eventually will be cancelled. After the trivial integrations are done in eq. (23), we have

$$\int -e^2G'\bar{U}\gamma_\mu(1-i\gamma_5)(K+k)^\mu\frac{1}{K^2-m_K^2}(2K-t)^\rho\frac{1}{(K-t)^2-m_K^2}\frac{1}{t^2-\lambda^2}(2K-t)_\rho$$

$$V\delta(K-p-k-q)d^4t \tag{24}$$

The difficulty with this expression is that we have $K^2-m_K^2$ in the denominator and this is zero for a free particle. To overcome this difficulty we introduce a damping factor $g(t)$ [7], replace e by

$eg(t) = e\int_{-\infty}^{\infty}G(\Gamma_o)e^{-i\Gamma_o t}d\Gamma_o$, set

$$\Gamma = (\Gamma^o,0,0,0)$$

and generalize by $eg(t)\to e\int G(\Gamma)e^{-i\Gamma t}d\Gamma$

with the normalization $g(0)=\int_{-\infty}^{\infty}G(\Gamma_o)d\Gamma_o = 1$

To do the complete calculation let us write down the T matrix explicitly,

$$\frac{1}{(2\pi)^4}\int (-iG')\overline{U}e^{-ipx_3}\,\gamma_\mu(1-i\gamma_5)(K''+k)^\mu\,\frac{1}{K''^2-m_K^2}\,e^{-iK''(x_2-x_3)}\,e^{-ikx_3}$$

$$(-ieg(t_1))\,\frac{1}{K'^2-m_K^2}\,e^{-iK'(x_1-x_2)}\,\frac{1}{t^2-\lambda^2}\,e^{-it(x_1-x_2)}(-ieg(t_2))$$

$$e^{iKx_1}(K''+K')_\rho\,(K'+K)^\rho\,Ve^{-iqx_3}\,d^4K''d^4K'd^4td^4x_1d^4x_2d^4x_3$$

$$= -\frac{1}{(2\pi)^{12}}\int e^2G'\overline{U}\gamma_\mu(1-i\gamma_5)(K''+k)^\mu\,\frac{e^{-iK''(x_2-x_3)}}{K''^2-m_K^2}\,\frac{e^{-it(x_1-x_2)}}{t^2-\lambda^2}\,\frac{e^{-iK'(x_1-x_2)}}{K'^2-m_K^2}$$

$$e^{-i\Gamma_1x}\,G(\Gamma_1)e^{-i\Gamma_2x}\,G(\Gamma_2)(K''+K')_\rho\,(K'+K)^\rho V\,e^{-i(p+k+q-K)x_3}\,d^4K''$$

$$d^4K'd^4td^4x_1d^4x_2d^4x_3d\Gamma_1d\Gamma_2$$

Carrying out all the trivial integrations, we have

$$-e^2G'\int \overline{U}\gamma_\mu(1-i\gamma_5)(K+k-\Gamma_1-\Gamma_2)^\mu\,G(\Gamma_1)G(\Gamma_2)d\Gamma_1d\Gamma_2\,\frac{1}{(K-\Gamma 1-\Gamma 2)^2-m_K^2}$$

$$\frac{1}{t^2-\lambda^2}\,\frac{1}{(K-t-\Gamma_1)^2-m_K^2}\,d^4t(2K-t-2\Gamma_1-\Gamma_2)_\rho(2K-t-\Gamma_1)^\rho\,\delta(K-\Gamma_1-\Gamma_2-p-k-q) \tag{25}$$

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Note that if $\Gamma_1=\Gamma_2=0$. We get back to the result of eq. (24). We may write eq. (25) as

$$-iG'\int \bar{U}\gamma_\mu(1-i\gamma_5)(K+k-\Gamma_1-\Gamma_2)^\mu d\Gamma_1 d\Gamma_2 G(\Gamma_1)G(\Gamma_2)\,\frac{1}{(K-\Gamma_1-\Gamma_2)^2-m_K^2}\,(2\pi)^4$$

$$\Sigma(\Gamma,K)\,\delta^4(K-\Gamma_1-\Gamma_2-p-k-q)V \tag{26}$$

Where Σ is the t dependent term

$$\Sigma(\Gamma,K) = -\frac{ie^2}{(2\pi)^4}\int d^4t\,\frac{1}{t^2-\lambda^2}\,\frac{1}{(k-t-\Gamma_1)^2-m_K^2}\,(2K-t-2\Gamma_1-\Gamma_2)_\rho(2K-t-\Gamma_1)^\rho$$

We expand Σ in powers of Γ and retain only terms of order Γ.

$$\Sigma(\Gamma,K) = -\frac{ie^2}{(2\pi)^4}\int d^4t\,\frac{1}{t^2-\lambda^2}\,\frac{1}{t^2-2Kt}\,(2K-t)^2 + \frac{ie^2}{(2\pi)^4}\int d^4t\,\frac{1}{t^2-\lambda^2}\,\cdot$$

$$\frac{1}{t^2-2Kt}\,(2K-t)(3\Gamma_1+\Gamma_2) - \frac{2ie^2}{(2\pi)^4}\int d^4t\,\frac{1}{t^2-\lambda^2}\,\frac{(2K-t)^2(K-t)_\rho\Gamma_1^\rho}{(t^2-2Kt)^2}$$

$$= I_1 + I_2 + I_3\,. \tag{27}$$

The first integral I_1 is the self-energy integral independent of Γ and is a function of K alone. This function must be a function of K^2 so that it be Lorentz Invariant. But $K^2=m_K^2$, hence the I_1 is a constant term and would be an infinite constant. It is a quadratically divergent term, which can always be cancelled by a term $-\delta m_K$.

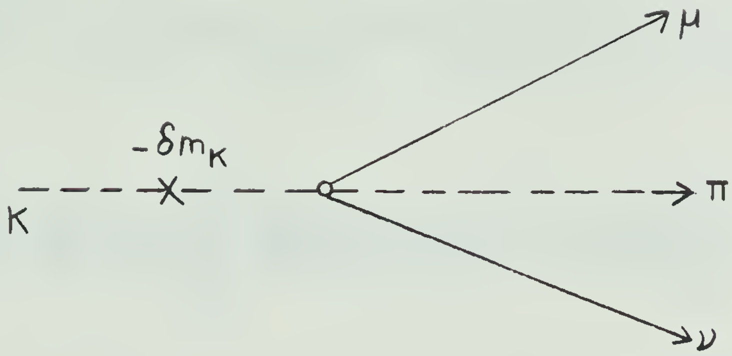


Fig. 4a

Therefore the integrals left for evaluation are I_2 and I_3. By using the Feynman cutoff method[8], we get

$$I_2 = -\frac{e^2}{16\pi^2}\left(\frac{3}{2}\log\frac{\Lambda^2}{m_K^2} + \frac{9}{4}\right)K_\rho(3\Gamma_1 + \Gamma_2)^\rho \tag{28}$$

$$I_3 = \frac{e^2}{16\pi^2}\left(\log\frac{\Lambda^2}{m_K^2} - 2\log\frac{m_K^2}{\lambda^2} + 6\right)2K_\rho\Gamma_1{}^\rho \tag{29}$$

Substitute these back into eq. (26) and eq. (27)

$$-iG'\int \bar{U}\gamma_\mu(1 - i\gamma_5)(K+k-\Gamma_1-\Gamma_2)^\rho V\, d\Gamma_1 d\Gamma_2 G(\Gamma_1)G(\Gamma_2)\,\frac{1}{(K-\Gamma_1-\Gamma_2)^2 - m_K^2}$$

$$\left[\frac{e^2}{16\pi^2}\left\{\left(-\frac{3}{2}\log\frac{\Lambda^2}{m_K^2} - \frac{9}{4}\right)K_\rho(3\Gamma_1+\Gamma_2)^\rho + 2\left(\log\frac{\Lambda^2}{m_K^2} - 2\log\frac{m_K^2}{\lambda^2} + 6\right)K_\rho\Gamma_1{}^\rho\right\}\right]$$

$$(2\pi)^4\delta(K-\Gamma_1-\Gamma_2-k-p-q) \tag{30}$$

We now symmetrize the factor Γ in eq. (17) by writing $\Gamma_1 \longrightarrow \frac{1}{2}\Gamma_1 + \Gamma_2$.

Then eq. (30) becomes

$$-iG' \int \bar{U}\gamma_\mu(1 - i\gamma_5)(K+k-\Gamma_1-\Gamma_2)^\mu V \, d\Gamma_1 d\Gamma_2 G(\Gamma_1)G(\Gamma_2) \frac{1}{-2K(\Gamma_1+\Gamma_2)}$$

$$\left[\frac{e^2}{16\pi^2}\left\{(-\frac{3}{2}\log\frac{\Lambda^2}{m_K^2} - \frac{9}{4})2K_\rho(\Gamma_1+\Gamma_2)^\rho + (\log\frac{\Lambda^2}{m_K^2} - 2\log\frac{m_K^2}{\lambda^2} + 6)\right.\right.$$

$$\left.\left. K_\rho(\Gamma_1+\Gamma_2)^\rho\right\}\right](2\pi)^4\delta(k-\Gamma_1-\Gamma_2-p-k-q) \qquad (31)$$

In the limit of $\Gamma_1, \Gamma_2 \to 0$, we get

$$T_1 = -iG'\bar{U}\gamma_\mu(1 - i\gamma_5)(K + k)^\mu\left[\frac{\alpha}{4\pi}(\log\frac{\Lambda^2}{\lambda^2} - \frac{3}{4})\right](2\pi)^4\delta(K-p-k-q) \qquad (32)$$

Where $\frac{e^2}{4\pi} = \alpha$ is the fine structure constant $\approx \frac{1}{137}$

Λ is the cut off mass used in Feynman regulator

λ is the fictitious photon mass, which is introduced to avert the intra-red divergence. As $\lambda \to 0$ the term $\log \lambda$ is infinite.

Now, we consider the correction due to the process of fig. 5. The matrix element corresponding to this process is

$$\int \bar{U}(-iG'e\gamma_\sigma)(1 - i\gamma_5)\frac{1}{(2\pi)^4}\frac{1}{(K'^2 - m_K^2)}\frac{-1}{(2\pi)^4}\frac{1}{t^2 - \lambda^2}(-ie)(K' + K)_\sigma V$$

$$(2\pi)^4\delta(K'+t-p-k-q)(2\pi)^4\delta(K-t-k')d^4K'd^4t \ .$$

After carrying out the K' integration with the help of the delta-function, we get

$$-e^2 G' \int \bar{U}\gamma_\sigma(1-i\gamma_5) \frac{1}{(K-t)^2 - m_K^2} \frac{1}{t^2-\lambda^2} (2K-t)^\sigma V\delta(K-p-k-q)d^4t \qquad (33)$$

The above integral will not give rise to an infra-red divergent term. By using the Feynman cutoff method, we get the result:

$$T_1' = \frac{i\alpha}{4\pi} G'\bar{U}\left(\frac{3}{2}\log\frac{\Lambda^2}{m_K^2} + \frac{9}{4}\right)\not{K}(1-i\gamma_5)V(2\pi)^4\delta(K-p-k-q) \qquad (34)$$

where Λ and α are the same quantities as defined earlier.

3.2 THE SELF-ENERGY OF THE MUON

Similar to the case of K meson self-energy, we have two types of muon self-energy graph, fig. 6 and fig. 7.

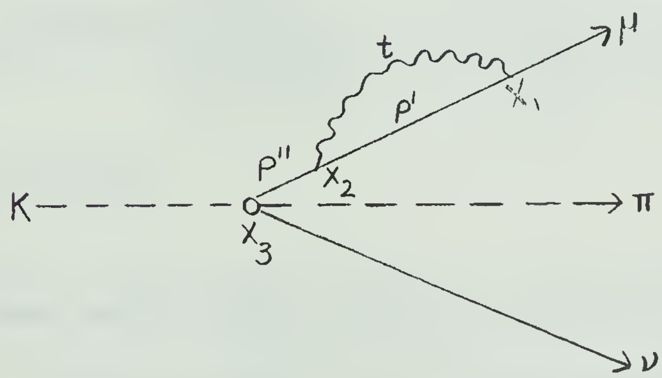


Fig. 6

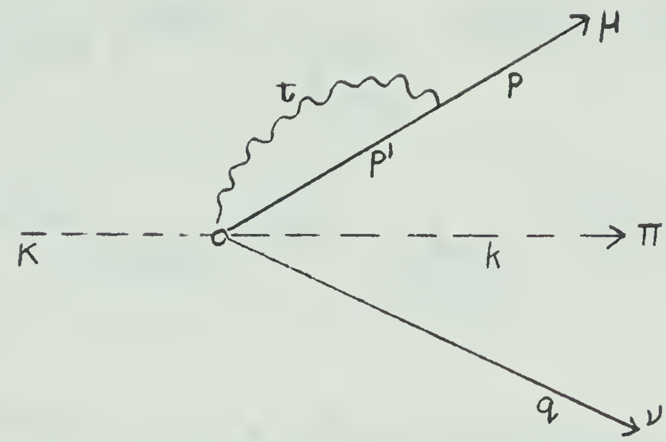


Fig. 7

The matrix element corresponding to the fig. 6 is:

$$T_2 = \int (-iG')\overline{U}(-ie\gamma^{\sigma}) \frac{i}{(2\pi)^4} \frac{1}{\not p' - m_\mu} (-ie\gamma_\sigma) \frac{i}{(2\pi)^4} \frac{1}{\not p'' - m_\mu} \frac{-i}{(2\pi)^4} \frac{1}{t^2 - \lambda^2}$$

$$\gamma_\mu(1-i\gamma_5)(K+k)^\mu V(2\pi)^4 \delta(K-p''-k-q)(2\pi)^4 \delta(p''-t-p')(2\pi)^4$$

$$\delta(p'+t-p)d^4p''d^4p'd^4t \ . \tag{35}$$

After carrying out the trivial integrations.

$$T_2 = \int -e^2 G'\overline{U}\gamma^{\sigma} \frac{\not p - \not t + m_\mu}{(p-t)^2 - m_\mu^2} \gamma_\sigma \frac{\not p + m_\mu}{p^2 - m_\mu^2} \frac{1}{t^2 - \lambda^2} \gamma_\mu (1-i\gamma_5)(K+k)^\mu V$$

$$\delta(K-p-k-q)d^4t \ . \tag{36}$$

For a free muon $p^2 = m_\mu^2$. Therefore the denominator is identically equal to zero. We may handle this in the same way as before, i.e. by replacing e by $eg(t) = e\int_{-\infty}^{\infty} G(\Gamma)e^{-i\Gamma t}d\Gamma$. Then the matrix element becomes

$$- \frac{1}{(2\pi)^{12}} \int e^2 G' \bar{U} \, e^{-ipx_1} \, \gamma^\sigma \frac{1}{\not{p} - m_\mu} e^{-ip'(x_2 - x_1)} \gamma_\sigma \frac{1}{\not{p}'' - m_\mu} e^{-ip''(x_3 - x_2)}$$

$$G(\Gamma_1) e^{-i\Gamma_1 x_1} \, G(\Gamma_2) e^{-i\Gamma_2 x_2} \, \frac{1}{t^2 - \lambda^2} \, e^{-it(x_2 - x_1)} \gamma_\mu (1 - i\gamma_5)(K + k)^\mu V$$

$$e^{i(K-k-q)x_3} d^4x_1 d^4x_2 d^4x_3 d^4p'' d^4p' d^4t d^4\Gamma_1 d^4\Gamma_2$$

$$= -iG' \int \bar{U} \sum(p,\Gamma) \frac{1}{\not{p} + \not{\Gamma}_1 + \not{\Gamma}_2 - m_\mu} \gamma_\mu (1 - i\gamma_5)(K+k)^\mu V (2\pi)^4 \delta(K-p-k-q)$$

$$G(\Gamma_1) G(\Gamma_2) d\Gamma_1 d\Gamma_2 \qquad (37)$$

Where $\sum(p,\Gamma)$ is the t dependent integral,

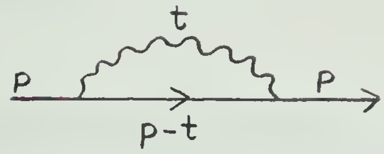
$$\sum(p,\Gamma) = \frac{-i}{(2\pi)^4} \int d^4t \, \gamma^\sigma \frac{1}{\not{p} - \not{t} + \not{\Gamma}_1 + m} \gamma_\sigma \frac{1}{t^2 - \lambda^2}$$

This is a logarithmically divergent term. As before, we expand $\sum$ in powers of Γ and retain terms of order Γ.

$$\sum(p,\Gamma) = - \frac{ie^2}{(2\pi)^4} \int d^4t \gamma^\sigma \frac{1}{\not{p} - \not{t} - m_\mu} \gamma_\sigma \frac{1}{t^2 - \lambda^2} + \frac{ie^2}{(2\pi)^4} \, d^4t \gamma^\sigma \frac{1}{\not{p} - \not{t} - m_\mu}$$

$$\gamma^\nu \frac{1}{\not{p} - \not{t} - m_\mu} \gamma_\sigma \frac{1}{t^2 - \lambda^2} \Gamma_\nu$$

$$= I_1' + I_2' \tag{38}$$

The term I_1' corresponds to the diagram 

which would be cancelled by the term $-\ \delta m_\mu$. The integral I_2' can be evaluated by standard methods as was done before.

$$I_2' = -\frac{2e^2}{16\pi^2}\left[\left(\tfrac{1}{2}\log\frac{\Lambda^2}{m_\mu^2}+\frac{5}{4}\right)\not{\Gamma} - \Gamma.p\left(\frac{1}{2m_\mu}-\frac{1}{2m_\mu}\log\frac{m_\mu^2}{\lambda^2}\right)\right]. \tag{39}$$

In which Λ, λ, α are the same quantities as defined earlier. Substituting the expression back into eq. (37), we obtain

$$T_2 = -iG'\int \overline{U}\left(-2\frac{\alpha}{4\pi}\right)\left[\left(\tfrac{1}{2}\log\frac{\Lambda^2}{m_\mu^2}+\frac{5}{4}\right)\not{\Gamma} - \Gamma.p\,\frac{1}{2m_\mu}\left(1-\log\frac{m_\mu^2}{\lambda^2}\right)\right]$$

$$\frac{1}{\not{p}+\not{\Gamma}_1+\not{\Gamma}_2-m_\mu}\,\gamma_\mu(1-i\gamma_5)(K+k)^\mu V(2\pi)^4\delta(K-p-k-q)G(\Gamma_1)G(\Gamma_2)d\Gamma_1 d\Gamma_2 \tag{40}$$

We now symmetrize the factor Γ in eq. (40) by writing $\Gamma = \frac{1}{2}(\Gamma_1+\Gamma_2)$ and recalling that $\not{p}-m_\mu$ operating on the free particle Dirac spinor gives zero, we may add the operator $\frac{1}{2}(\not{p}-m_\mu)$ to the first term in the bracket in eq. (40).

$$-iG'\int \overline{U}\left(-\frac{\alpha}{2\pi}\right)\left[\frac{1}{2}\log\frac{\Lambda^2}{m^2}+\frac{5}{4}\right]\tfrac{1}{2}(\not{p}+\not{\Gamma}_1+\not{\Gamma}_2-m_\mu)\,\frac{1}{(\not{p}+\not{\Gamma}_1+\not{\Gamma}_2-m_\mu)}$$

$$\gamma_\mu(1-\gamma_5)(K+k)^\mu V(2\pi)^4\delta(K-p-k-q)G(\Gamma_1)G(\Gamma_2)d\Gamma_1 d\Gamma_2 \tag{41}$$

As $\Gamma_1, \Gamma_2 \longrightarrow$ o, the above expression becomes

$$-iG'\overline{U}\left[-\frac{\alpha}{4\pi}\left(\tfrac{1}{2}\log\frac{\Lambda^2}{m_\mu^2}+\frac{5}{4}\right)\right]\gamma_\mu(1-i\gamma_5)(K+k)^\mu V(2\pi)^4\delta(K-p-k-q) \qquad (42)$$

We rewrite the second term as follows:

$$-iG'\int\overline{U}\left(\frac{\alpha}{2\pi}\right)\left[\frac{1}{2m_\mu}\left(\log\frac{m_\mu^2}{\lambda^2}-1\right)\right]\frac{\not{p}+\not{\Gamma}_1+\not{\Gamma}_2+m_\mu}{2p.(\ \ _1+\ _2)}\,\tfrac{1}{2}p.(\Gamma_1+\Gamma_2)\gamma_\mu(1-i\gamma_5)$$

$$(K+k)^\mu V(2\pi)^4\delta(K-p-k-q)$$

In the limit $\Gamma_1, \Gamma_2 \longrightarrow 0$, this reduces to

$$-iG'\overline{U}\left[\frac{\alpha}{4\pi}\left(\log\frac{m_\mu^2}{\lambda^2}-1\right)\right]\gamma_\mu(1-i\gamma_5)(K+k)^\mu V(2\pi)^4\delta(K-p-k-q) \qquad (43)$$

Adding eq. (42) and eq. (43), we have

$$T_2 = -iG'\overline{U}\left[\frac{\alpha}{4\pi}\left(\log\frac{m_\mu^2}{\lambda^2}-\tfrac{1}{2}\log\frac{\Lambda^2}{m_\mu^2}-\frac{9}{4}\right)\right]\gamma_\mu(1-i\gamma_5)(K+k)^\mu V(2\pi)^4\delta(k-p-k-q) \qquad (44)$$

The matrix element for the graph 7 is given as follows:

$$\int\overline{U}(-ie\gamma^\sigma)\frac{i}{(2\pi)^4}\cdot\frac{1}{\not{p}-m_\mu}(-ieG'\gamma_\sigma)\,\frac{-1}{(2\pi)^4}\,\frac{1}{t^2-\lambda^2}\,(1-i\gamma_5)V(2\pi)^4$$

$$\delta(k-p'-q-k-t)(2\pi)^4\delta(p-p'-t)d^4t\,d^4p' \qquad (45)$$

$$= -eG' \int \overline{U}\gamma^{\sigma} \frac{\not{p}-\not{t}+m}{(p-t)^2-m^2} \gamma_{\sigma} \frac{1}{t^2-\lambda^2} (1-i\gamma_5)V\delta(K-p-k-q)d^4t$$

The above integral does not contribute any infrared divergent term. After integrating with the help of Feynman method, we get

$$T_2' = \frac{i\alpha}{4\pi} G'\overline{U}(3\log \frac{\Lambda^2}{m_\mu^2} + \frac{3}{2})m_\mu(1-i\gamma_5)V(2\pi)^4\delta(K-p-k-q) \tag{46}$$

3.3 THE VERTEX CORRECTION

The Feynman diagram corresponding to the vertex correction is given in fig. 8.

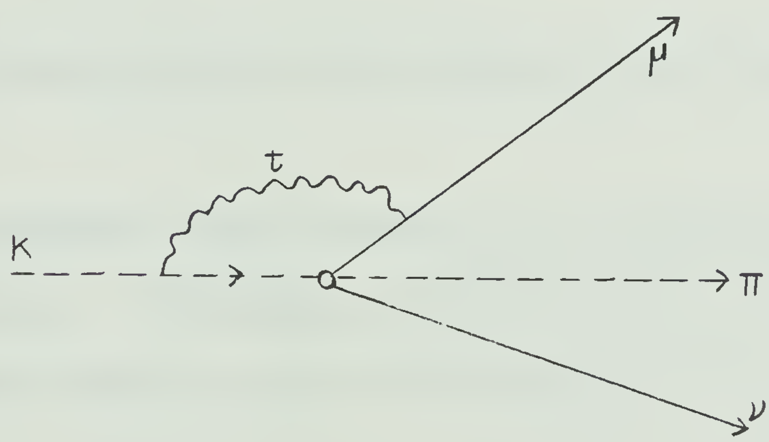


Fig. 8

The expression for the matrix element for this process is as follows:

$$\int (-iG')\overline{U}(-ie\gamma^{\sigma}) \frac{i}{(2\pi)^4} \frac{1}{\not{p}'-m_\mu} \gamma_\mu (1-i\gamma_5)(K'+k)^\mu \frac{i}{(2\pi)^4} \frac{1}{K'^2-m_K^2} (-ie)$$

$$(K+K')_\sigma \frac{-i}{(2\pi)^4} \frac{1}{t^2} V(2\pi)^4 \delta(K'-p'-k-q)(2\pi)^4 \delta(K'+t-K)(2\pi)^4$$

$$\delta(p'+t-p)d^4p'd^4K'd^4t$$

$$= -\int e^2 G'\bar{U}\gamma^\sigma \frac{1}{\not{p}-\not{t}-m_\mu} \gamma_\mu (1-i\gamma_5)(K-t+k)^\mu \frac{1}{(K-t)^2-m_K^2} (2K-t)_\sigma \frac{1}{t^2} (K-p-k-q)d^4t$$

$$= (-i)(2\pi)^4 G'\bar{U}\mathcal{J}(K,p)V\delta(K-p-k-q) \qquad (47)$$

Where $\mathcal{J}(K,p)$ is the p-dependent integral whose explicit form is

$$\mathcal{J}(K,p) = \frac{-i}{(2\pi)^4}\int \gamma^\sigma \frac{1}{\not{p}-\not{t}-m_\mu} \gamma_\mu (k-t+k)^\mu (1-i\gamma_5) \frac{1}{(K-t)^2 - m_K^2} (2K-t)_\sigma \frac{1}{t^2} d^4t \qquad (48)$$

The numerator in eq. (48) can be arranged as

$$(2\not{K}-\not{t})(\not{p}-\not{t}+m)(\not{K}-\not{t}+\not{k}) = (4K.p-2\not{K}\not{t}-\not{t}\not{p}-\not{t}m_\mu+\not{t}^2)(\not{K}+\not{k}) - (4K.p\not{t}-2\not{K}\not{t}^2-\not{t}\not{p}\not{t}-\not{t}^2m_\mu+\not{t}^3) \qquad (49)$$

The numerator being of order t^3 the integral $\mathcal{J}(K,p)$ is logarithmically divergent. Introducing a regulator mass Λ and carrying out the standard Feynman Parametrization the denominator of $\mathcal{J}(K,p)$ can be written as

$$\int \frac{1}{t^2} \frac{1}{t^2-2pt} \frac{1}{t^2-2Kt} d^4t = -6 \int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 \frac{u(1-u)dudt}{\left[(t-k_y u)^2-k_y^2u^2-L(1-u)\right]^4} \qquad (50)$$

where $k_y = p + G(1-y)$. Substituting both eq. (50) and eq. (49) into eq. (48), we have the expression,

$$\mathcal{J}(K,p) = \frac{-i}{(2\pi)^4}\left\{-6\int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \int dt \,\frac{(4K.p-2\not{K}\not{t}-\not{t}\not{p}-\not{t}m_\mu+\not{t}^2)(\not{K}+\not{k})}{\left[(t-k_y u)^2-k_y^2u^2-L(1-u)\right]^4}\right.$$

$$\left.+6\int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \int dt \,\frac{4K.p\not{t}-2\not{K}\not{t}^2-\not{t}\not{p}\not{t}-\not{t}^2m_\mu+\not{t}^3}{\left[(t-k_y u)^2-k_y^2u^2-L(1-u)\right]^4}\right\}.(1-i\gamma_5)$$

On Shifting the origin, by replacing t by $t+k_y u$, and dropping all terms of odd order in t in the numerator, we obtain

$$\mathcal{J}(K,p) = \frac{-i}{(2\pi)^4}\left\{\left[-6\int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \int dt \,\frac{(4K.p-\not{k}_y\not{p}u-m_\mu\not{k}_y u-2\not{K}\not{k}_y u+k_y^2u^2)}{\left[t^2-k_y^2u^2-L(1-u)\right]^4}\right.\right.$$

$$-6\int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \int dt \,\frac{t^2}{\left[t^2-k_y^2u^2-L(1-u)\right]^4}\quad(\not{K}+\not{k})$$

$$+6\int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \int dt \,\frac{4K.p\not{k}_y u-2\not{K}\not{k}_y^2u^2-\not{k}_y\not{p}\not{k}_y u^2-m_\mu\not{k}_y^2u^2+\not{k}_y^3u^3}{\left[t^2-k_y^2u^2-L(1-u)\right]^4}$$

$$+6\int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \int dt \frac{-2\cancel{K}t^2+\frac{1}{2}\cancel{p}t^2-m_\mu t^2}{\left[t^2-k_y^2u^2-L(1-u)\right]^4} \Bigg\} (1-i\gamma_5)$$

$$= \frac{-i}{(2\pi)^4} \Bigg\{\Bigg[-\int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \frac{i\pi^2 4k.p}{\left[(k_y^2u^2+L(1-u)\right]^2}$$

$$+2i\pi^2 \int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \frac{(k_y.p+\cancel{K}\cancel{k}_y)u}{\left[k_y^2u^2+L(1-u)\right]^2}$$

$$-i\pi^2 \int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \frac{k_y^2u^2}{\left[k_y^2u^2+L(1-u)\right]^2}$$

$$+2i\pi^2 \int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \frac{1}{\left[k_y^2u^2+L(1-u)\right]}\Bigg](K+k)$$

$$+i\pi^2 \int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \frac{4K.p\cancel{k}_y u}{\left[k_y^2u^2+L(1-u)\right]^2}$$

$$-i\pi^2 \int_{\lambda^2}^{\Lambda^2} dL \int_0^1 dy \int_0^1 u(1-u)du \frac{(2\cancel{K}\cancel{k}_y^2+\cancel{k}_y\cancel{p}\cancel{k}_y+m_\mu \cancel{k}_y^2)u^2}{\left[k_y^2u^2+L(1-u)\right]^2}$$

$$+i\pi^2\int_{\lambda^2}^{\Lambda^2} dL\int_0^1 dy\int_0^1 u(1-u)du\;\frac{\not{k}_y^3u^3}{\left[k_y^2u^2+L(1-u)\right]^2}$$

$$+2i\pi^2\int_{\lambda^2}^{\Lambda^2} dL\int_0^1 dy\int_0^1 u(1-u)du\;\frac{2\not{K}+\frac{1}{2}m_\mu}{\left[k_y^2u^2+L(1-u)\right]}\Bigg\}(1-i\gamma_5) \qquad (51)$$

In eq. (51) only the first integral diverges when $\lambda = 0$. Therefore for the rest of the integrals we may take $\lambda = 0$.

$$\mathcal{J}(K,p) = \frac{ie^2}{16\pi^2}\Bigg\{\Bigg[-2K.p\int_0^1 \frac{1}{k_y^2}\log\frac{k_y^2}{\lambda^2}\,dy + 2\int_0^1 dy(k_y.p+\not{K}\not{k}_y)\,\frac{1}{k_y^2} - \tfrac{1}{2} +$$

$$+\int_0^1 dy\,\log\frac{\Lambda^2}{k_y^2} - \tfrac{1}{2}\Bigg](\not{K}+\not{k}) + 4K.p\int_0^1 dy\,\frac{\not{k}_y}{k_y^2} -$$

$$-\tfrac{1}{2}\int_0^1 dy(2\not{K}\not{k}_y^2+\not{k}_y\not{p}\not{k}_y+m_\mu\not{k}_y^2)1/k_y^2 + \frac{1}{3}\int_0^1 dy\,\frac{\not{k}_y^3}{k_y^2} +$$

$$+\;2(2\not{K}+\tfrac{1}{2}m_\mu)\left[\int_0^1 dy\left(\log\frac{\Lambda^2}{k_y^2}\right)-\tfrac{1}{4}\right]\Bigg\}(1-i\gamma_5) \qquad (52)$$

In the above expression, there are five integrals that need to be

evaluated.

$$J_o = \int_o^1 \frac{1}{k_y^2} \log \frac{k_y^2}{\lambda^2}\, dy$$

$$J_1 = \int_o^1 \frac{k_{y\sigma} dy}{k_y^2}$$

$$J_2 = \int_o^1 \log k_y^2 dy$$

$$J_3 = \int_o^1 \frac{k_{y\sigma} k_{y\tau} dy}{k_y^2}$$

$$J_4 = \int_o^1 \frac{k_{y\sigma}^3}{k_y^2}\, dy$$

To evaluate these integrals we make the substitution y=1-z, then

$$k_y^2 = G^2(z^2 + 2\, \frac{p.G}{G^2}\, z + \frac{m_\mu^2}{G^2})$$

If we take $\frac{p.G}{G^2} = a$ and $\sqrt{(p.G/G^2)^2 - (m^2/G^2)} = b$, then

$$k_y^2 = G^2(z+a+b)(z+a-b)$$

When one evaluates all those integrals in terms of these variables, one gets

$$J_o = \int_o^1 \frac{dz}{G^2(z+a+b)(z+a-b)} \log\left[\frac{G^2}{\lambda^2}(z+a+b)(z+a-b)\right]$$

$$= F_o \log\frac{G^2}{\lambda^2} + F_o' \tag{53}$$

Where $$F_o = \frac{1}{2bG^2}\left[\log\left(\frac{1+a-b}{1+a+b}\right)\left(\frac{a+b}{a-b}\right)\right] \tag{54}$$

$$F_o' = -\frac{1}{2bG^2}\left\{\frac{1}{2}\left[\log(1+a-b)\right]^2 - \frac{1}{2}\left[\log(a-b)\right]^2 - \frac{1}{2}\left[\log(1+a+b)\right]^2\right.$$

$$+\frac{1}{2}\left[\log(a+b)\right]^2 - \log(1+a+b)\log(1+a-b) +$$

$$+\log(a+b)\log(a-b) + L\left(-\frac{1+a-b}{2b}\right) - L\left(-\frac{a+b}{2b}\right) +$$

$$\left.+\log 2b . \log\left(\frac{1+a-b}{a+b}\right)\right\} \tag{55}$$

The L functions in the eq. (55) are the Spence functions, which are defined as

$$L(t) = \int_o^t \log(1-x)\frac{dx}{x}$$

Near the end point of the muon (or pion) spectrum, the term designated as F_o' will not give rise to a divergence. By making the same substituting as that for J_o, we can evaluate the rest of J's.

$$J_1 = \int_0^1 \frac{p_\sigma + G_\sigma z}{G^2(z+a+b)(z+a-b)}\,dz = p_\sigma F_o + G_\sigma F_1$$

Where $$F_1 = \frac{1}{2bG^2}\left[-(a-b)\log\left(\frac{1+a-b}{a-b}\right) + (a+b)\log\left(\frac{1+a+b}{a+b}\right)\right] \tag{56}$$

$$J_2 = \int_0^1 \log\left[G^2(z+a+b)(z+a-b)\right]dz = \log G^2 + F_2$$

Where $$F_2 = \left[(a-b)\log\frac{1+a-b}{a-b} + (a+b)\log\frac{1+a+b}{a+b} + \log(1+a+b)(1+a-b)\right] \tag{57}$$

$$J_3 = \int_0^1 \frac{p_\sigma p_\tau + (G_\sigma p_\tau + p_\sigma G_\tau)z + G_\sigma G_\tau z^2}{G^2(z+a+b)(z+a-b)}\,dz = p_\sigma p_\tau F_o + (G_\sigma p_\tau + p_\sigma G_\tau)F_1 + G_\sigma G_\tau F_3$$

Where $$F_3 = \frac{-1}{2bG^2}\left[(a+b)^2\log\left(\frac{1+a+b}{a+b}\right) - (a-b)\log\left(\frac{1+a-b}{a-b}\right) + 3b\right] \tag{58}$$

$$J_4 = \int_0^1 k_{y\sigma}^3/k_y^2\,dy = p_\sigma + \tfrac{1}{2}G_\sigma \tag{59}$$

Substituting these results back into eq. (52), we have

$$\mathcal{I}(K,p) = \frac{e^2}{16\pi^2}\left\{\left[-2K.p(F_o\log\frac{G^2}{\lambda^2} + F_o') + 2(p^2F_o + p.GF_1) + 2(\not{K}\not{p}F_o + \not{K}\not{G}F_1)\right.\right.$$

$$\left. - \tfrac{1}{4} + \log\frac{\Lambda^2}{G^2} - F_2\right](\not{K}+\not{k}) + 4K.p(\not{p}F_o + \not{G}F_1) + \not{K} - m_\mu +$$

$$+ \tfrac{1}{2}\not{p}^3F_o + \not{G}m_\mu^2F_1 + \tfrac{1}{2}\not{G}\not{p}\not{G}F_3 + \frac{1}{3}F_4 + (4\not{K}+m_\mu)\log\frac{\Lambda^2}{G^2} -$$

$$\left. - (4\not{K}+m_\mu)F_2\right\}(1-i\gamma_5) \tag{60}$$

If we are working in the frame where K meson is at rest, then $K = (m_K,0,0,0)$, and $\overline{U}\not{p} = m_\mu\overline{U}$. So we can write eq. (60) as follows:

$$\mathcal{I}(K,p) = \frac{e^2}{16\pi^2}\left\{\left[-2m_KE(F_o\log\frac{G^2}{\lambda^2} + F_o') + 2(m_\mu^2F_o + m_KEF_1 - m^2F_1) +\right.\right.$$

$$\left. + 4m_kE(F_o - F_1) + 2m_K^2F_1 + \log\frac{\Lambda^2}{G^2} - \tfrac{1}{4} - F_2\right](\not{K}+\not{k})$$

$$+ \left[(-\frac{5}{6}m_\mu - 2m_\mu m_K^2 + 4m_\mu m_KE) + 4m_\mu m_KE(F_o - F_1) + \tfrac{1}{2}F_om_\mu^3 - m_\mu^2F_1 -\right.$$

$$\left. - \tfrac{1}{2}F_3m_\mu m_K^2 - m_\mu\log\frac{\Lambda^2}{G^2} + 4m_\mu F_2 + \tfrac{1}{2}m_\mu^3F_3\right] + \left[-2m_\mu^2 + 4m_KEF_1 +\right.$$

$$\left.\left. + m_KEF_3 - m_\mu^2 + 4\log\frac{\Lambda^2}{G^2} - 4F_2\right]\not{K} - \not{K}\not{q}\right\}(1-i\gamma_5) \tag{61}$$

Finally the matrix element for the vertex case is

$$T_3 = (-i)G'\overline{U}\left\{\frac{\alpha}{4\pi}\left[(2m_KEF_o\log\lambda^2 + f_1')(\not{K}+\not{k}) + f_2' + f_3'\not{K} + \not{K}\not{q}\right]\right\}$$

$$(1-i\gamma_5)V(2\pi)^4\delta(K-p-k-q) \tag{62}$$

Where $f_1' = -2m_K E(F_o \log G^2 + F_o') + 2(m_\mu^2 F_o - m_K E F_1 - m_\mu^2 F_1) + 4m_K E(F_o - F_1) +$

$$+ 2m_K^2 + \log \frac{\Lambda^2}{G^2} - F_2 - \frac{1}{4}. \tag{63}$$

$f_2' = 4m_\mu^2 m_K E - \frac{5}{6} m_\mu - 2m_\mu m_K^2 + 4m_\mu m_K E(F_o - F_1) + \frac{1}{2}F_o m_\mu^3 - m_\mu^3 F_1 -$

$$- \tfrac{1}{2}m_\mu m_K^2 F_3 + \tfrac{1}{2}m_\mu^3 F_3 - m_\mu \log \frac{\Lambda^2}{G^2} + 4m_\mu F_2. \tag{64}$$

$$f_3' = -2m_\mu^2 + 4m_K E F_1 + m_K E F_3 - m_\mu^2 + 4\log \frac{\Lambda^2}{G^2} - 4F_2 \tag{65}$$

In eq. (62) the log λ term is the infrared term that we had before. The function F_o', F_o, F_1, F_2 and F_3 are functions of G^2 throughout the physical region $(m_\mu - m_K)^2 \geq G^2 \geq m_\pi^2$. At the end-point, G tends to m_π, and these functions vary slowly as the end-point of the muon (or pion) spectrum is approached. It is justifiable to regard the functions of f_1', f_2' and f_3' as constant when compared to the divergent terms.

Adding all the virtual photon corrections to the uncorrected decay amplitude T_o, one has

$$T_{vp} = T_o + T_1 + T_1' + T_2 + T_2' + T_3 \tag{66}$$

where $$T_o = (-i)G'\overline{U}\gamma_\mu(1-i\gamma_5)(K+k)^\mu V(2\pi)^4\delta(K-p-k-q) \tag{67}$$

which corresponds to the diagram

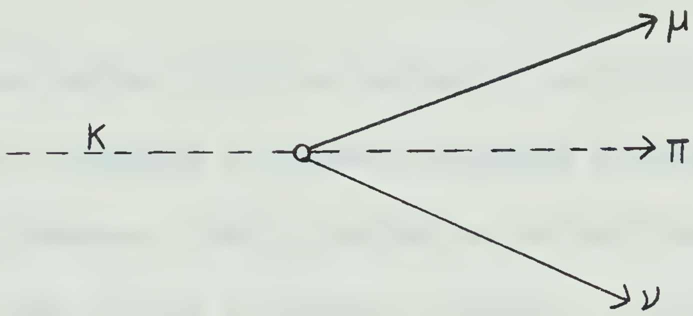


Fig. 9

Putting the eq. (32), eq. (34), eq. (44), eq. (46) and eq. (62) into eq. (66), we have the expression for the amplitude corrected by the virtual photon processes.

$$T_{vp} = (-i)G'\overline{U}\gamma_\mu(1-i\gamma_5)\left\{1 + \frac{\alpha}{4\pi}\left(\left[-(2-2m_K E F_o)\log\lambda^2 - f_1\right](K+k)\right.\right.$$

$$\left.\left. + f_2 + f_3 K^\mu + K^\mu\gamma_\sigma q^\sigma\right)\right\} V(2\pi)^4\delta(k-p-k-q)$$

$$= T_o\left\{1 + \frac{\alpha}{4\pi}\left[-2-2m_K E F_o)\log\lambda^2 - f_1\right]\right\}$$

$$+ \left[\frac{\alpha}{4\pi}(-i)G'\overline{U}(f_2 + f_3\not{K} + \not{K}\not{q})\gamma(1-i\gamma_5)V(2\pi)^4\delta(K-p-k-q)\right] \qquad (68)$$

where we have written

$$f_1 = f_1' + \tfrac{1}{2}\log\Lambda^2 + \tfrac{2}{3}\log^2_\mu - 3 \qquad (69)$$

$$f_2 = f_2' - \left[3\log\frac{\Lambda^2}{m_\mu^2} - \frac{3}{2}\right] m_\mu \qquad (70)$$

$$f_3 = f_3' - \frac{3}{2}\log\frac{\Lambda^2}{m_\mu^2} - \frac{9}{4} \tag{71}$$

In the expression (68) the infrared term $(2-2m_K EF_o)\log\lambda^2$ will be cancelled by a similar infrared divergent term arising from the real photon emission process. The calculation of the real photon emission correction is done in the following Chapter. The ultra-violet terms in eq. (68) can be written as $\log\frac{\Lambda^2}{m_\mu^2}$ or $\log\frac{\Lambda^2}{m_K^2}$, then these terms will not give a large contribution to the spectrum for any plausible value of Λ.

CHAPTER IV REAL PHOTON CORRECTION (INNER BREMSSTRAHLUNG)

There are two types of innerbremsstrahlung graph, fig. (10) and fig. (11), which will contribute to the correction.*

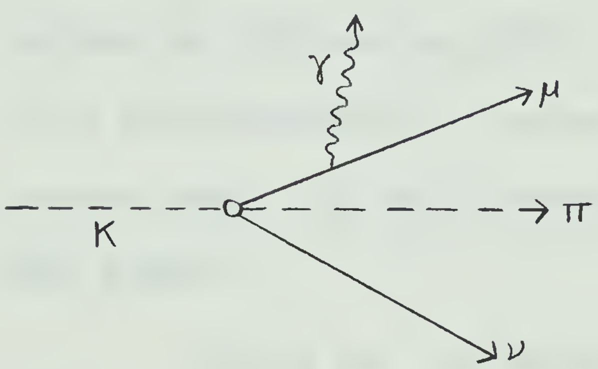


Fig. 10

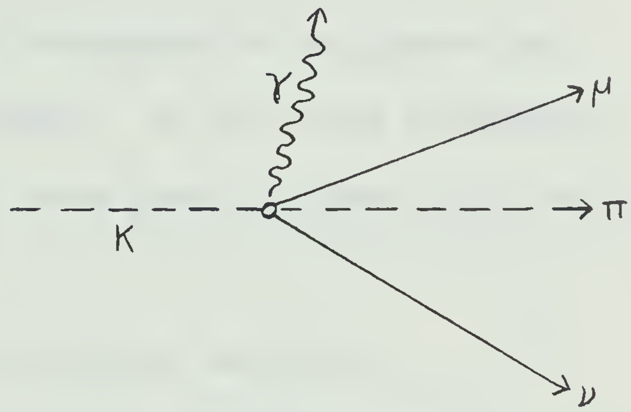


Fig. 11

The matrix element corresponding to fig. (10) is

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* The innerbremsstrahlung graph of fig. (12) will not five any contribution to the correction, as we work in the frame where K meson is at rest. This can be seen as follows: The vertex factor for the photon emitted from the K meson line will be $(2K-k').\varepsilon$, where ε is the polarization vector of the real photon and is orthogonal to k' i.e. $k'.\varepsilon=0$. We can choose a gauge where ε has only the space components. Therefore in the time-like K frame $K.\varepsilon=0$. Hence the vertex factor vanishes.

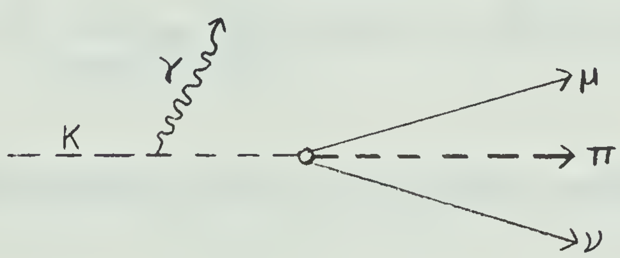

Fig. 12

$$T'_{IB} = (-iG)\overline{U}(-ie\gamma^{\rho})\frac{\varepsilon^{\lambda}_{\rho}}{(2\pi)^{3/2}}\,\frac{1}{(\not{p}-\not{k}')-m_{\mu}}\,\gamma_{\mu}(1-i\gamma_5)(K+k)^{\mu}V(2\pi)^4\delta(K-k'-p-k-q) \tag{72}$$

The matrix element corresponding to fig. (11) is

$$T''_{IB} = (-iG')\overline{U}(e\gamma^{\mu})\,\frac{\varepsilon^{\lambda}_{\mu}}{(2\pi)^{3/2}}\,(1-i\gamma_5)V(2\pi)^4\delta(K-p-k-q-k') \tag{73}$$

In both eq. (72) and eq. (73), k' is the 4-momentum of the photon and ε^{λ} if the polarization 4-vector of the photon. k' and $\mathcal{E}$ are orthogonal to each other. If one is working in the frame where K meson is at rest, then $K.\mathcal{E}=0$.

Adding the two matrix elements together, we have

$$T_{IB}=T'_{IB}+T''_{IB}=(-iG')\overline{U}\,\frac{e}{(2\pi)^{3/2}}\left[\frac{\not{\mathcal{E}}(\not{p}-\not{k}'+m_{\mu})}{(p-k)^2-m^2}\,(K+K)+\not{\mathcal{E}}\right](1-i\gamma_5)V$$

$$(2\pi)^4\delta(K-k'-p-k-q) \tag{74}$$

By making use of the relations $\overline{U}\,\not{p} = m_{\mu}\overline{U}$

and $\not{\mathcal{E}}\,\not{p} = 2p.\mathcal{E} - \not{p}\not{\mathcal{E}}$

we can simplify eq. (74) as follows:

$$T_{IB} = (-iG')\overline{U}\,\frac{e}{(2\pi)^{3/2}}\left[\frac{\mathcal{E}.p}{p.k'}\,(\not{K}+\not{k}) + \not{\mathcal{E}}\right](1-i\gamma_5)V(2\pi)^4\delta(K-p-k-q-k') \tag{75}$$

The decay probability due to the photon emission alone is given by

$$\int \left|T_{IB}\right|^2 \frac{d^3p}{2E}\frac{d^3k}{2\omega}\frac{d^3q}{2\epsilon}\frac{d^3k'}{2\omega'}$$

$$= \int \left|T'_{IB}\right|^2 \frac{d^3p}{2E}\frac{d^3k}{2}\frac{d^3q}{2}\frac{d^3k'}{2\ '} + \int \left|T''_{IB}\right|^2 \frac{d^3p}{2E}\frac{d^3k}{2\omega}\frac{d^3q}{2\epsilon}\frac{d^3k'}{2\omega'}$$

$$+ \int 2\mathrm{Re}\left|\chi_{IB}\right| \frac{d^3p}{2E}\frac{d^3k}{2\ }\frac{d^3q}{2\epsilon}\frac{d^3k'}{2\omega'} \qquad (76)$$

where χ_{IB} is the cross-product term.

$$\chi_{IB} = \sum_{s\lambda} \frac{e^2 G'^2}{(2\pi)^3}\left[\overline{U}(\frac{p.\mathcal{E}}{p.k'})(\not{K}+\not{k})(1-i\gamma_5)V\right]\left[\overline{V}(1-i\gamma_5)\not{\mathcal{E}}U\right](2\pi)^4\delta(K-k'-p-k-q) \qquad (77)$$

Since we are not concerned about the spin of the neutretto and the polarization of the photon, we have to sum χ_{IB} over the final spin states and over the polarization vectors of the photon. And the same applies to $\left|T'_{IB}\right|^2$ and $\left|T''_{IB}\right|^2$.

Let us calculate the first term in the eq. (76). We can write it as

$$\int \left|T'_{IB}\right|^2 \frac{d^3p}{2E}\frac{d^3k}{2\omega}\frac{d^3q}{2\epsilon}\frac{d^3k'}{2\omega'} = \int \left|T_o\right|^2 \frac{d^3p}{2E}\frac{d^3k}{2\epsilon}\frac{d^3q}{2\omega}\ \frac{e^2}{(2\pi)^3}\frac{\left|p.\mathcal{E}\right|^2}{\left|p.k'\right|^2}\frac{d^3k'}{2\omega'} \qquad (78)$$

Where T_o is the uncorrected decay amplitude.

$$\left|T_o\right|^2 = (2\pi)^4 G'^2\left|U(\not{K}+\not{k})(1-i\gamma_5)V\right|^2\delta(K-p-k-q) \qquad (79)$$

Summing over the polarization vectors of photon[9], we have

$$\sum_{\lambda}' \frac{|p.\varepsilon|^2}{|p.k'|^2} = \frac{\frac{1}{2}\left[\bar{p}^2-(\bar{p}.\bar{k}'^2/\omega'^2)\right]}{(E\omega'-\bar{p}.\bar{k}')^2} = \frac{\beta^2}{2}\,\frac{(1-\mho^2\cos^2\theta)}{(1-\mho\beta\cos\theta)^2} \tag{80}$$

In where $\mho = \frac{k'}{\omega'}$ corresponds to the velocity of the photon.

$\beta = |\bar{p}|/E$, which is the velocity of the muon.

θ is the angle between the momentum of the muon and that of the photon.

To integrate $\int \frac{|p.\varepsilon|^2}{|p.k'|^2}\frac{d^3k'}{2\omega'}$, we take the direction of the $\bar{p}$ as the z-axis direction and do the integration in the spherical coordinates (Appendix B)[10]. We have

$$\int \frac{|p.\varepsilon|^2}{|p.k'|^2}\frac{d^3k'}{2\omega'} = 2\pi(I_o + C(\beta)) \tag{81}$$

with

$$I_o = \tfrac{1}{2}\beta^2\int_{-1}^{+1}\left(\log\frac{k'_{max}}{\lambda}\right)\frac{1-x^2}{(1-\beta x)^2} \tag{82}$$

and

$$C(\beta) = \tfrac{1}{2}\Big\{-4B(\beta)(\log 2-2) + \frac{2}{\beta}\left[L(\beta) - L(-\beta)\right] - \frac{1}{\beta}\left[L\left(\frac{1+\beta}{2}\right) - L\left(\frac{1-\beta}{2}\right)\right] -$$

$$- \tfrac{1}{2}\,\frac{1}{\beta}\log\frac{1-\beta}{1+\beta}\left(\log\frac{1-\beta^2}{4} + 2\right) + 2\Big\} \tag{83}$$

In eq. (83) the L functions are Spence functions as defined earlier and $B(\beta)$ is defined as

$$B(\beta) = 1 + \frac{1}{2\beta} \log \frac{1-\beta}{1+\beta} \tag{84}$$

$B(\beta)$ can be shown to be identical to the term $(1-m_K EF_o)$ in eq. (68). If we write

$$a = \frac{p.G}{G^2} = \frac{(m_K/E) - 1 + \beta^2}{(m_K/E)^2 - 2(m_K/E) + 1 - \beta^2}$$

$$b = \sqrt{\left(\frac{p.G}{G^2}\right)^2 - \frac{m_\mu^2}{G^2}} = \frac{\beta m_K/E}{(m_K/E)^2 - 2(m_K/E) + 1 - \beta^2}$$

then
$$-m_K EF_o = -m_K E\left[\frac{1}{2bG^2} \log \left(\frac{1+a-b}{1+a+b}\right)\left(\frac{a+b}{a-b}\right)\right]$$

$$= \frac{1}{2\beta} \log \frac{1-\beta}{1+\beta}$$

Near the end-point of the muon spectrum, $G^2 = m_\pi^2 + 2m_\pi \delta E$, the photon is not allowed to carry away too large an amount of energy. The maximum value of $\overline{k}'$ can be calculated from the relation

$$(G - k')^2 = (q + k)^2 \tag{85}$$

The maximum photon energy occurs when $(q+k)^2$ is a minimum, i.e. $(m_\pi + m_\nu)^2$. In the rest frame of the K meson $G = (m_K - E, \overline{p})$, we get

$$\overline{k}'_{max} = \frac{m_K \delta E - m_\pi m_\nu}{m_K - E - \overline{p}\cos\theta} \tag{86}$$

Therefore we can write I_o as

$$I_o = -2B(\beta)\log \frac{m_K \delta E - m_\pi m_\nu}{\lambda} + D(\beta) \tag{87}$$

where $$D(\beta) = -\tfrac{1}{2}\beta^2 \int_{-1}^{+1} \log(m_K - E - \overline{p}\cos\theta)\, \frac{1-x^2}{(1-\beta x)^2}\, dx \tag{88}$$

$D(\beta)$ is a well behaved quantity. The explicit evaluation of it is not necessary, because when we take the ratio of the spectra it will drop out.

$$\int |T'_{IB}|^2 = \int |T_o|^2 \frac{d^3p}{2E}\frac{d^3k}{2\omega}\frac{d^3q}{2\epsilon}\left[\frac{\alpha}{\pi}\left\{-2B(\beta)\log\frac{m_K\delta E - m_\pi m_\nu}{\lambda} + C(\beta) + D(\beta)\right\}\right] \tag{89}$$

Now, let us evaluate the cross product term i.e. eq. (77).

$$\sum_{\lambda_1 s} = \left[\overline{U}\left(\frac{p.\varepsilon^\lambda}{p.k'}\right)(\not{k}+\not{k})(1-\gamma_5)V\right]\left[\overline{V}(1+i\gamma_5)\not{\varepsilon}^\lambda U\right]$$

$$= \sum_\lambda \mathrm{Tr}\left[\frac{1}{2m_\mu m_\nu}\left(\frac{p.\varepsilon^\lambda}{p.k'}\right)(\not{p}+m_\mu)(\not{K}+\not{k})\not{q}\not{\varepsilon}^\lambda(1+i\gamma_5)\right]$$

$$= \sum_\lambda \frac{2}{m_\mu m_\nu}\left(\frac{p.\varepsilon^\lambda}{p.k'}\right)\Big[(p.K)(q.\varepsilon^\lambda) - (p.q)(K.\varepsilon^\lambda) + (p.\varepsilon^\lambda)(K.q) +$$

$$+(p.k)(q.\mathcal{E}^{\lambda})-(p.q)(k.\mathcal{E}^{\lambda})+(p.\mathcal{E}^{\lambda})(k.q)\Big] \quad *$$

$$=\sum_{\lambda}\frac{2}{m_{\mu}m_{\nu}}\left(\frac{p.\mathcal{E}^{\lambda}}{p.k'}\right)\Big[2(p.K)(q.\mathcal{E}^{\lambda})+2(p.\mathcal{E}^{\lambda})(K.q)-(p.p)(q.\mathcal{E}^{\lambda})$$

$$-(p.k')(q.\mathcal{E}^{\lambda})-(p.\mathcal{E}^{\lambda})(q.k')\Big] \tag{90}$$

In which we have made use of the equality K+k = 2K-p-q-k' and the conditions $K.\mathcal{E}^{\lambda}=0$ and $k'.\mathcal{E}^{\lambda}=0$;
and have neglected the term which is multiplied by $(q.q) = m_{\nu}^2$ since it is quadratic in m_{ν} and small compared to the other terms. Substituting eq. (90) back in the expression for χ_{IB}, we obtain

$$\chi_{IB} = \frac{e^2G'^2}{(2\pi)^3}\frac{2}{m_{\mu}m_{\nu}}\sum_{\lambda}\frac{p.\mathcal{E}^{\lambda}}{p.k'}\Big\{(q.\mathcal{E}^{\lambda})\Big[2(p.K)-(p.p)-(p.k')\Big] + (p.\mathcal{E}^{\lambda})$$

$$-\Big[2(K.q)-(q.k')\Big]\Big\}(2\pi)^4\delta(K-k'-p-k-q) \tag{91}$$

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* We have omitted the terms like $\not{p}\not{K}\not{q}\not{\mathcal{E}} = -4\mathcal{E}_{\mu\nu\rho\sigma}\, p^{\mu}K^{\nu}q^{\rho}\mathcal{E}^{\sigma}$. Note that $\mathcal{E}^{\sigma}$ has only the space components,

i.e. $\mathcal{E}^{o} = 0$, $\mathcal{E}^{i} \neq 0$ $i = 1,2,3$.

Expanding the tensor terms one will get terms like $\mathcal{E}^{i}$....
In any cartesian frame one will have $\mathcal{E}^{i}$ with angles θ, ψ for $\bar{\mathcal{E}}$.
Therefore, one will get quantities like

$$\sum\cos\theta, \qquad \sum\sin\theta\cos\psi, \text{ and } \qquad \sum\sin\theta\sin\psi.$$

But the average of all of them is zero.

$$\int_0^{\pi}\cos\theta d\theta = 0, \quad \int_0^{\pi}\int_0^{2\pi}\sin\theta\cos\psi d\theta d\psi = 0, \quad \int_0^{\pi}\int_0^{2\pi}\sin\theta\sin\psi d\theta d\psi = 0.$$

Hence, these terms will not contribute.

When summing and averaging over the polarization vectors of the photon, we have

$$(p.\varepsilon^{\lambda})^2 = \tfrac{1}{2}\left[|\overline{p}|^2 - \frac{p.k'^{\;2}}{\overline{k}'^2}\right]$$

$$(p.\varepsilon^{\lambda})(q.\varepsilon^{\lambda}) = \tfrac{1}{2}\left[\overline{p}.\overline{q} - \frac{(\overline{p}.\overline{k}')(\overline{q}.\overline{k}')}{\overline{k}'^2}\right]$$

And we also have the identities:

$$p.k' = E\omega' - \overline{p}.\overline{k}' \qquad q.k' = \varepsilon\omega' - \overline{q}.\overline{k}'$$

$$p.K = m_K E \qquad p.p = m_\mu^2 \qquad q.K = m_K\epsilon$$

Putting all these quantities into eq. (91), we get

$$\chi_{IB} = \frac{e^2G'^2}{(2\pi)^3}\frac{1}{m_\mu m_\nu}\left[(2m_KE - m_\mu^2 - E\omega' + \overline{p}.\overline{k}')\left(|\overline{p}|^2 - \frac{|\overline{p}.\overline{k}'|^2}{\overline{k}'^2}\right) + \right.$$

$$\left. + (2m_K\epsilon - \epsilon\omega' + \overline{q}.\overline{k}')\left(\overline{p}.\overline{q} - \frac{(\overline{p}.\overline{k}')(\overline{q}.\overline{k}')}{\overline{k}'^2}\right)\right]\frac{1}{p.k'}(2\pi)^4\delta(K-k'-p-k-q) \tag{92}$$

Let the angle between $\overline{p}$ and $\overline{k}'$ be θ, the angle between $\overline{q}$ and $\overline{k}'$ be θ' and the angle between $\overline{p}$ and $\overline{q}$ be φ. If we take the direction of $\overline{p}$ as the direction of the z-axis and $\overline{q}$ to be in the x-z plane, then the relationship between θ and θ' is given by

$$\cos\theta' = \cos\theta\cos\varphi + \sin\theta\cos\psi\sin\varphi \tag{93}$$

In eq. (93) ψ is the azimuthal angle.

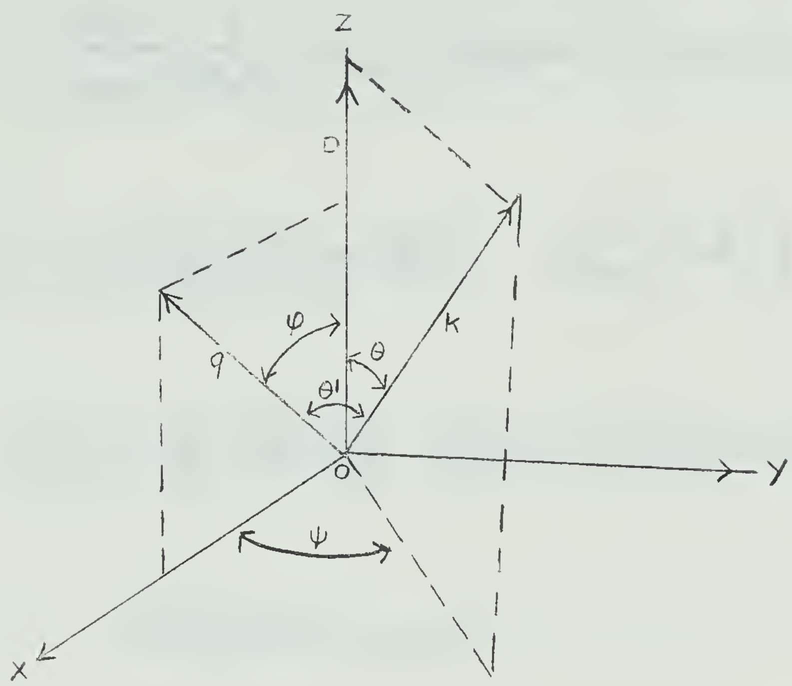

Fig. 13

Recall that in calculating the decay probability, we have to integrate χ_{IB} with respect to $\frac{d^3p}{2E}\,\frac{d^3k}{2\omega}\,\frac{d^3q}{2\epsilon}\,\frac{d^3k'}{2\omega'}$. For the moment, we leave out the integration with respect to the first three variables, and do only the integration with respect to $\frac{d^3k'}{2\omega'}$.

$$\int |\chi|_{IB}\,\frac{d^3k'}{2\omega'} = \int \frac{e^2G'^2}{(2\pi)^3}\,\frac{1}{m_\mu m_\nu}\left\{\frac{1}{(E\omega' - |\bar{p}||\bar{k}'|\cos\theta)}\left[(2m_KE - m^2 - E\ ' + |\bar{p}||\bar{k}'|\cos\theta)\right]\right.$$

$$\left[\bar{p}.\bar{q} - |\bar{p}||\bar{q}|(\cos^2\theta\cos\varphi + \sin\theta\,\cos\theta\,\cos\psi\,\sin\varphi)\right] +$$

$$\left. + \left[2m_K\epsilon - \epsilon\omega' + |\bar{q}||\bar{k}'|(\cos\theta\cos\varphi + \sin\theta\cos\psi\sin\varphi)\right.\ |\bar{p}|^2\sin^2\theta\right\}$$

$$\frac{|\bar{k}'|^2 d\bar{k}' d\psi d\cos\theta}{2\omega'}\,(2\pi)^4\delta(K - k' - p - k - q)$$

$$= \frac{e^2 G'^2}{(2\pi)^4} \frac{1}{2m_\mu m_\nu} \Bigg[\left|\bar{k}'_{max}\right| \left[(2m_K E - m^2)\left|\bar{q}\right| \cos\varphi - 2m_K \epsilon \left|\bar{p}\right| \right] \left[\log \frac{1+\beta}{1-\beta} - \right.$$

$$\left. - \frac{1}{\beta^2} \left(6\beta + \log \frac{1+\beta}{1-\beta}\right) \right] - \tfrac{1}{2} k'^2_{max} \Bigg\{ \bar{p}.\bar{q} \left[\frac{4}{3} - \frac{1}{\beta^2} \left(-2\beta + \log \frac{1+\beta}{1-\beta}\right) \right.$$

$$\left. - \frac{1}{\beta^4} \left(\log \frac{1+\beta}{1-\beta} - \frac{2}{3}\beta^3 - 2 \right) \right] + \left|\bar{p}\right| \epsilon \left[\log \frac{1+\beta}{1-\beta} - \frac{1}{\beta^2} \left(6\beta + \log \frac{1+\beta}{1-\beta}\right) \right] \Bigg\} \Bigg]$$

$$(2\pi)^4 \delta(K - k' - p - k - q) \tag{94}$$

We can see from eq. (94) that the correction due to the cross-term is well behaved for all muon momenta. Let us write the cross-term as:

$$\int |\chi|_{IB} \frac{d^3p}{2E} \frac{d^3k}{2\omega} \frac{d^3q}{2\epsilon} \frac{d^3k'}{2\omega'} = \int A(\beta) \frac{d^3p}{2E} \frac{d^3k}{2\omega} \frac{d^3q}{2\epsilon} \tag{95}$$

The evaluation of the integral $\int \left|T''_{IB}\right|^2$ is much simpler than the two previous cases.

$$\left|T''_{IB}\right|^2 = \frac{e^2 G'^2}{(2\pi)^3} \sum_{s\lambda} \left[\bar{U} \not{\varepsilon} (1 - i\gamma_5) V \right]^2 (2\pi)^4 \delta(K - k' - p - k - q)$$

$$= \frac{e^2 G'^2}{(2\pi)^3} \frac{1}{2m_\mu m_\nu} \sum_\lambda \mathrm{Tr} \left[(\not{p} + m_\mu) \not{\varepsilon} \not{q} (1 + i\gamma_5) \right] (2\pi)^4 \delta(K - k' - p - k - q)$$

$$= \frac{e^2 G'^2}{(2\pi)^3} \frac{2}{m_\mu m_\nu} \sum_\lambda \left[(p.\varepsilon^\lambda)(q.\varepsilon^\lambda) - (p.q)\varepsilon^{\lambda 2} \right] (2\pi)^4 \delta(K - k' - p - k - q)$$

$$= \frac{e^2 G'^2}{(2\pi)^3} \frac{1}{m_\mu m_\nu}\left[\bar{p}.\bar{q} - \frac{(p.k')(q.k')}{k'^2} + p.q\right](2\pi)^4 \delta(K-k'-p-k-q)$$

$$\int |T''_{IB}|^2 \frac{d^3p}{2E} \frac{d^3k}{2\omega} \frac{d^3q}{2\epsilon} \frac{d^3k'}{2\omega'}$$

$$= \frac{e^2 G'^2}{(2\pi)^3} \frac{1}{m_\mu m_\nu} \int \frac{d^3p}{2E} \frac{d^3k}{2\omega} \frac{d^3q}{2\epsilon} \left[2E\epsilon - |\bar{p}||\bar{q}|(1+\cos\theta\cos\theta')\right]$$

$$\frac{\bar{k}'^2 d\bar{k}' d\psi d\cos\theta}{2\omega'} (2\pi)^4 \delta(K-k'-p-k-q)$$

$$= \int A'' \frac{d^3p}{2E} \frac{d^3k}{2\omega} \frac{d^3q}{2\epsilon} \tag{96}$$

where $A'' = \frac{e^2 G'^2}{(2\pi)^2} \frac{1}{m_\mu m_\nu} |k'_{max}|^2 (4E\epsilon - \frac{8}{3}\bar{p}.\bar{q})(2\pi)^4 \delta(K-k'-p-k-q)$

Finally, the radiative correction due to the real photon emission process is

$$\int \left\{ |T_o|^2 \left[\frac{\alpha}{\pi}\left(-2B(\beta)\log\frac{m_K \delta E - m_\mu m_\nu}{\lambda} + C(\beta) + D(\beta)\right)\right] + A'' + A(\beta) \right\} \frac{d^3p}{2E} \frac{d^3k}{2\omega} \frac{d^3q}{2\epsilon} \tag{97}$$

Note that the infrared divergent term which arises here is equal but opposite in sign to that from the virtual photon process corrections (c.f. eq. (68) Chapter III).

CHAPTER V THE SPECTRA OF MUON AND PION

5.1 THE SPECTRUM OF THE MUON

When evaluating the muon spectrum we carry out all integrations except the one over the muon momentum in the expression for the decay probability. Moreover, the virtual photon correction and the real photon correction contribute incoherently to the spectrum, because they lead to final states with different number of particles.

The radiative corrections from the virtual photon processes to the muon spectrum are

$$dP'(E,m_\nu) = \int \left|T_{vp}\right|^2 \frac{d^3k}{2\omega}\frac{d^3q}{2\epsilon}$$

From eq. (68)

$$T_{vp} = (-iG')\overline{U}\left\{1 + \frac{\alpha}{4\pi}\left[-4B(\beta)\log\lambda - f_1\right](\not{K}+\not{k}) + \frac{\alpha}{4\pi}(f_2+f_3\not{K}+\not{K}\not{q})\right\}$$

$$(1-i\gamma_5)V(2\pi)^4\delta(K-p-k-q)$$

where we have put $2(1-m_kEF_o) = 2B(\beta)$. Now, we take the square of the modulus of T_{vp} and sum over the spin states of neutretto, we get

$$\left|T_{vp}\right|^2 = \left|T_o\right|^2\left[1+\frac{2\alpha}{\pi}\left(-B(\beta)\log\lambda - \tfrac{1}{4}f_1\right)\right] + R \qquad (98)$$

Where T_o is the uncorrected decay amplitude, which was defined earlier (c.f. eq. (67) Chapter III), and R is the contribution from the cross-product terms.

$$R = 2\mathrm{Re}\sum_s G'^2 \frac{\alpha}{4\pi}\left[\overline{U}(2\not{K}-m_\mu)(1-i\gamma_5)V + \overline{U}(-m_\nu)(1+i\gamma_5)V\right]$$

$$\left[\overline{V}(1+i\gamma_5)(f_2+f_3\not{K})U + \overline{V}(1-i\gamma_5)\not{K}m_\nu U\right](2\pi)^4\delta(K-k-p-q) \tag{99}$$

In both eq. (98) and eq. (99), terms of order higher than e^2 have been neglected. Expanding the expression for R, we have

$$R = 2\mathrm{Re}\sum_s \frac{\alpha G'^2}{4\pi}\Big\{\left[\overline{U}(2\not{K}-m_\mu)(1-i\gamma_5)V\right]\left[\overline{V}(1+i\gamma_5)(f_2+f_3\not{K})U\right] +$$

$$+ \left[\overline{U}(2\not{K}-m_\mu)(1-i\gamma_5)V\right]\left[\overline{V}(1-i\gamma_5)\not{K}m_\nu U\right] +$$

$$+ \left[\overline{U}(-m_\nu)(1+i\gamma_5)V\right]\left[\overline{V}(1+i\gamma_5)(f_2+f_3\not{K})U\right] +$$

$$+ \left[\overline{U}(-m_\nu)(1+i\gamma_5)V\right]\left[\overline{V}(1-i\gamma_5)\not{K}m_\nu U\right]\Big\}(2\pi)^4\delta(K-p-k-q)$$

$$= R_1 + R_2 + R_3 + R_4$$

And $$R_1 = \frac{G'^2\alpha}{2\pi}\mathrm{Re}\sum_s\left[\overline{U}(2\not{K}-m_\mu)(1-i\gamma_5)V\right]\left[\overline{V}(1+i\gamma_5)(f_2+f_3\not{K})U\right]$$

$$(2\pi)^4\delta(K-p-k-q)$$

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$$= \frac{2\alpha G'^2}{\pi m_\mu m_\nu}\Big[2f_3 m_K^2(E\ \ + \overline{p}.\overline{q}) + (2m_\mu f_2 - m_\mu^2 f_3) m_K \epsilon\ -$$

$$-m_\mu f_2(E\epsilon - \overline{p}.\overline{q})\Big](2\pi)^4 \delta(K-p-k-q) \tag{100}$$

The rest of the R's are of order m_ν^2 in comparison to R. Since m_ν^2 is small, we may neglect them and keep only R_1.

Hence, the muon spectrum with virtual photon correction will be

$$dP'(E,m\ \ ) = \int\left\{\left|T_o\right|^2\left[1+\frac{2\alpha}{\pi}\left(-B(\beta)\log\lambda - \tfrac{1}{4}f_1\right)\right] + R\right\}\frac{d^3q}{2\epsilon}\,\frac{d^3k}{2\omega} \tag{101}$$

The real photon correction to the muon spectrum is

$$dP''(E,m_\nu) = \int\left|T_{IB}\right|^2 \frac{d^3q}{2\epsilon}\,\frac{d^3k}{2\omega}\,\frac{d^3k'}{2\omega'}$$

It follows from eq. (97), that

$$dP''(E,m_\nu) = \iint\left[\left|T_o\right|^2 \frac{\alpha}{\pi}\left\{-2B(\beta)\log\frac{m_K\delta E - m_\pi m_\nu}{\lambda} + C(\beta) + D(\beta)\right\} + A'' + A(\beta)\right]$$

$$\frac{d^3q}{2\epsilon}\,\frac{d^3k}{2\omega} \tag{102}$$

The final spectrum after radiative corrections is

$$dP(E,m_\nu) = dP'(E,m_\nu) + dP''(E,m_\nu)$$

$$= \int\left\{\left|T_o\right|^2\left[1-\frac{\alpha}{\pi}(2B(\beta)\log(m_K\delta E - m_\pi m_\nu)-C(\beta)-D(\beta)+\tfrac{1}{2}f_1)\right]+R+\right.$$

$$\left.+A+A''(\beta)\right\}\frac{d^3q}{2\varepsilon}\,\frac{d^3k}{2\omega} \tag{103}$$

In eq. (103) the explicit infrared divergent terms have cancelled but there remains a residual divergent term $\log(m_K\delta E-m_\pi m_\nu)$ which tends to $-\infty$ as the end-point of the spectrum is approached. This reflects the lack of cancellation of infrared divergences.

The muon spectrum is now obtained by carrying out the $\bar{q}$ and $\bar{k}$ integrations. One of these integrations, say the $\bar{q}$-integration, is trivial because of the presence of momentum conserving delta-function. The range of the last integration variable is limited by energy momentum conservation. We need not know this range for the calculation of the muon spectrum because the important terms in eq. (103) are all $\bar{k}$-independent and we shall only be interested in the ratio $dP(E,m_\nu)/dP(E,0)$. However for the sake of completeness we shall digress here for a moment and evaluate the range of energy available to the pion when the muon energy is fixed. The calculation of the range of muon energy for a fixed value of pion energy proceeds in an exactly analogous fashion.

In the K-meson-rest-frame,

$$k_{\substack{max\\min}} = \left|\bar{p}\right| \pm \left|\bar{q}\right| \tag{104}$$

or
$$k^2_{\substack{max \\ min}} = |\bar{p}|^2 + |\bar{q}|^2 \pm |\bar{p}||\bar{q}| \tag{105}$$

We shall carry out the calculation for the case $m_\nu=0$ only. Then $|\bar{q}|=\epsilon$, the energy of the neutretto. By energy conservation

$$\epsilon = m_K - E - \omega \tag{106}$$

Dropping the indices 'max' and 'min' and substituting eq. (106) in eq. (105) we get

$$2\omega(m_K - E \pm \bar{p}) = (m_K - E \pm \bar{p})^2 + m_\pi^2 \tag{107}$$

or
$$\frac{m_K-E+|\bar{p}|}{2} + \frac{m_\pi^2}{2(m_K-E+|\bar{p}|)} \leq \omega \leq \frac{m_K-E-|\bar{p}|}{2} + \frac{m_\pi^2}{2(m_K-E-|\bar{p}|)} \tag{107}$$

In an analogous fashion for every pion energy ω, the energy range of the muon is

$$\frac{m_K-\omega+|\bar{k}|}{2} + \frac{m_\mu^2}{2(m_K-\omega+|\bar{k}|)} \leq E \leq \frac{m_K-\omega-|\bar{k}|}{2} + \frac{m_\mu^2}{2(m_K-\omega-|\bar{k}|)} \tag{108}$$

We can now show from eq. (107) that if the muon is observed near its maximum energy then the pion energy, ω, has a very limited range close to its maximum value. The absolute energy maxima are

$$E_{max} = \frac{m_K^2 - m_\pi^2 + m_\mu^2}{2m_K} \tag{109}$$

$$\omega_{max} = \frac{m_K^2 - m_\mu^2 + m_\pi^2}{2m_K} \tag{110}$$

$$\beta_{max} = \frac{m_K^2 - m_\pi^2 - m_\mu^2}{m_K^2 - m_\pi^2 + m_\mu^2} \tag{111}$$

Suppose we observe muons with energy

$$E = E_{max} - \delta E \tag{112}$$

and momentum

$$\bar{p} = \bar{p}_{max} - \delta p \tag{113}$$

with

$$\delta\bar{p} = \delta E/\beta_{max} \tag{114}$$

Substituting eq. (112) - eq. (114) in eq. (107) we find that the minimum pion energy is

$$\omega = \omega_{max} - \tfrac{1}{2}\delta E\left(\frac{1}{\beta_{max}} - 1\right) \tag{115}$$

where ω_{max} is given by eq. (110) and $1/\beta_{max} > 1$. Thus the pion carries away almost its maximum energy when the muon does so too.

We will now return to the discussion of the muon spectrum. The R term in eq. (103) contains terms f_2 and f_3. These terms as mentioned in Chapter III, are small compared with the terms which diverge at the end-point of the muon spectrum. The A terms as we can see from

eq. (94) and eq. (96) of Chapter IV, are well behaved and do not give rise to any divergent term. We may drop all these terms in the expression of the muon-spectrum near the end-point. Therefore the muon-spectrum near the end-point is

$$dP(E_m,m_\nu) = \int |T_o|^2 \left[1 - \frac{\alpha}{\pi}\left\{2B(\beta_m)\log(m_K\delta E - m_\pi m_\nu) - C(\beta) - D(\beta) + \tfrac{1}{2}f_1\right\}\right]\frac{d^3q}{2\omega}\,\frac{d^3k}{2\epsilon} \tag{116}$$

The ratio $dP(E_m,m_\nu)/dP(E_m,0)$ can now be written as follows:

$$dP(E_m,m_\nu)/dP(E_m,0)$$

$$= \frac{\int |T_o|^2_{m_\nu\neq 0}\left[1 - \frac{\alpha}{\pi}\left\{2B(\beta_m)\log(m_K\delta E - m_\pi m_\nu) - C(\beta_m) - D(\beta_m) + \tfrac{1}{2}f_1\right\}\right]\frac{d^3q}{2\epsilon}\,\frac{d^3k}{2\omega}}{\int |T_o|^2_{m_\nu=0}\left[1 - \frac{\alpha}{\pi}\left\{2B(\beta_m)\log(m_K\delta E) - C(\beta_m) - D(\beta_m) + \tfrac{1}{2}f_1\right\}\right]\frac{d^3q}{2\epsilon}\,\frac{d^3k}{2\omega}}$$

$$= \frac{dP_o(E_m,m_\nu)\left[1 - \frac{\alpha}{\pi}\left\{2B(\beta_m)\log(m_K\delta E - m_\pi m_\nu) - C(\beta_m) - D(\beta_m) + \tfrac{1}{2}f_1\right\}\right]}{dP_o(E_m,0)\left[1 - \frac{\alpha}{\pi}\left\{2B(\beta_m)\log(m_K\delta E) - C(\beta_m) - D(\beta_m) + \tfrac{1}{2}f_1\right\}\right]} \tag{117}$$

where $dP_o(E_m,m_\nu)$ and $dP_o(E_m,0)$ are the uncorrected muon spectrum with $m_\nu\neq 0$ and that with $m_\nu=0$ respectively. Their expressions were given in eq. (6) and eq. (7) of Chapter I. The value of β_m with $m_\nu\neq 0$ differs from β_m with $m_\nu=0$ by terms of order of m_ν. We may take the functions $C(\beta_m)$, $D(\beta_m)$ and f_1 to be equal in both cases. Then, we can write eq. (117) as

$$\frac{dP(E_m,m_\nu)}{dP(E_m,0)} = \frac{dP_o(E_m,m_\nu)}{dP_o(E_m,0)}\left[1 - \frac{\alpha}{\pi}2B(\beta_m)\left\{\log(m_K\delta E - m_\pi m\ ) - \log(m_K\delta E)\right\}\right] \tag{118}$$

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The ratio of the uncorrected spectra $dP_o(E_m,m_\nu)/dP_o(E_m,0)$ is the same as that in eq. (12) of Chapter I. Hence, eq. (118) becomes

$$\frac{dP(E_m,m_\nu)}{dP(E_m,0)} = \left[1 - \frac{m_\pi^2 m_\nu^2}{m_K^2(\delta E)^2}\right]^{\frac{1}{2}} \left[1 - \frac{\alpha}{\pi}\, 2B(\beta_m)\log\left(1 - \frac{m_\pi m_\nu}{m_K \delta E}\right)\right] \tag{119}$$

5.2 THE SPECTRUM OF THE PION

In evaluating the full pion spectrum (i.e. the spectrum for all pion energies) we run into difficulties. These difficulties have their origin in the fact that the radiative correction terms are functions of the muon momentum and an integration over the muon spectrum has to be done to get the pion spectrum. Though a numerical integration is feasible, it is extremely difficult, if at all possible, to get an analytic form for the spectrum. One can get around this dificulty if one is interested in only the end-point (and its neighbourhood) of the pion spectrum. In this case the muon energy is close to its maximum as was shown in the discussion on energy ranges in the section 5.1 on the muon spectrum. The p-dependent integrand can be treated as a constant with $\overline{p}$ replaced by $\overline{p}_{max}$.

The calculation proceeds in an analogous fashion to that of the muon spectrum. We only have to express the maximum soft photon energy in terms of the pion energy rather than the muon energy. The precise relation is

$$\overline{k}'_{max} = \frac{m_K\delta\omega - m_\mu m_\nu}{m_K - \omega + \overline{k}\cos\xi} \qquad (120)$$

where $\cos\xi = \overline{k}.\overline{k}'$

and $\delta\omega = \omega_{max} - \omega$

The pion spectrum will then be (analogous to eq. (116))

$$dP(\omega_m, m_\nu) = \int \left|T_o\right|^2 \left[1 - \frac{\alpha}{\pi}\left\{2B(\beta_m)\log(m_K\delta\omega - m_\mu m_\nu) - C(\beta_m) - D(\beta_m) + \tfrac{1}{2}f_1\right\}\right] \frac{d^3q}{2\epsilon}\,\frac{d^3p}{2E} \tag{121}$$

By following the same reasoning as that for the muon spectrum we obtain

$$\frac{dP(\omega_m, m)}{dP(\omega_m, 0)} = \left[1 - \frac{m_\mu^2 m_\nu^2}{m_K^2(\delta\omega)^2}\right]^{\frac{1}{2}} \left[1 - \frac{\alpha}{\pi}\, 2B(\beta_m)\log(1 - \frac{m_\mu^2 m_\nu^2}{m_K\delta\omega})\right] \tag{122}$$

5.3 ENERGY CORRELATION BETWEEN MUON AND PION

Let us consider the phase space integral with radiative corrections,

$$\int |T_o|^2 \left[1 - \frac{\alpha}{\pi}\left\{2B(\beta)\log(m_K\delta E - m_\pi m_\nu) - C(\beta) - D(\beta) + \tfrac{1}{2}f_1\right\}\right] \frac{d^3p}{2E}\,\frac{d^3k}{2\omega}\,\frac{d^3q}{2\epsilon} \tag{123}$$

In eq. (123) we have dropped the term R, A and A" because we are going to look at the correlation at the high energy end, where the contribution from these terms is insignificant. Eq. (123) can be written as

$$\int |T_o|^2 \left[1 - \frac{\alpha}{\pi}\left\{2B(\beta)\log(m_K\delta E - m_\pi m_\nu) - C(\beta) - D(\beta) + \tfrac{1}{2}f_1\right\}\right] \frac{d^3p}{2E}\,\frac{d^3k}{2\omega}$$

$$d^4q\,\delta(q_o^2 - \epsilon^2)\,\delta^4(G-k-q)$$

$$= \int |T_o|^2 \left[1 - \frac{\alpha}{\pi}\left\{2B(\beta)\log(m_K\delta E - m_\pi m_\nu) - C(\beta) - D(B) + \tfrac{1}{2}f_1\right\}\right] \frac{8\pi^2\bar{k}^2\bar{p}^2 d\bar{k}d\bar{p}}{2E\omega|\bar{p}||\bar{k}|}$$

$$\delta\left[(m_K-p-k)^2 - m_\nu^2\right] d\cos\eta$$

$$= \int |T_o|^2 \left[1 - \frac{\alpha}{\pi}\left\{2B(\beta)\log(m_K\delta E - m_\pi m_\nu) - C(\beta) - D(\beta) + \tfrac{1}{2}f_1\right\}\right] \frac{2\pi^2\bar{k}^2\bar{p}^2 d\bar{k}d\bar{p}}{2E\omega|\bar{p}||\bar{k}|}$$

$$\delta\left[\cos\eta - \frac{(m_K-E-k_o)^2 - \bar{p}^2 - \bar{k}^2 - m^2}{2|\bar{p}||\bar{k}|}\right] d\cos\eta \tag{124}$$

where η is the angle between pion and muon momenta. The integration over the angle $\int \delta[\cos\eta - (\ldots)]\,d\cos\eta$ is equal to 1. Eq. (124) becomes

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$$\int \pi^2 |T_o|^2 \left[1 - \frac{\alpha}{\pi}\left\{2B(\beta)\log(m_K\delta E - m_\pi m_\nu) - C(\beta) - D(\beta) + \tfrac{1}{2}f_1\right\}\right] dE d\omega$$

$$= \int \rho(E,\omega) dE d\omega \tag{125}$$

where $\rho(E,\omega)$ is the energy correlation between the pion and muon for the particular model Hamiltonian we have chosen.

$$|T_o|^2 = \sum_s G'^2 |\overline{U}(\not{K}+\not{k})(1-i\gamma_5)V|^2$$

$$= \sum_s G'^2 \left\{ |\overline{U}(2\not{K}-m_\mu)(1-i\gamma_5)V|^2 + 2\mathrm{Re}\left[\overline{U}(2\not{K}-m_\mu)(1-i\gamma_5)V\right]\right.$$

$$\left.\left[\overline{V}(1-i\gamma_5)(-m_\nu)U\right] + m_\nu^2 |\overline{U}(1+i\gamma_5)V|^2 \right\} \tag{126}$$

The last 2 terms in the eq. (126) when compared with the first term are of order m_ν^2 and m_ν^4 respectively and so are negligible.

$$|T_o|^2 = G'^2 \frac{1}{m_\mu m_\nu} \left[8(E\epsilon + \overline{p}.\overline{q})m_K^2 - 8m_\mu^2 m_K \epsilon + 2m_\mu^2(E\epsilon - \overline{p}.\overline{q})\right] \tag{127}$$

Since $(p+q)^2 = (K-k)^2$, if the K meson decays at rest, then

$$E\epsilon - \overline{p}.\overline{q} = \tfrac{1}{2}(m_K^2 - 2m_K\omega + m_\pi^2 - m_\mu^2 - m_\nu^2)$$

$$= m_K(\omega_{max} - \omega) \tag{128}$$

In deriving eq. (128) we have made use of eq. (110) and have neglected terms of order m_ν^2. Substitute eq. (128) into eq. (127), we get

$$|T_o|^2 = G'^2 \frac{1}{m_\mu m_\nu} \left\{ 8\left[2E\epsilon - m_K(\omega_{max} - \omega)\right] - 8m_K m_\mu^2 \epsilon + 2m_\mu^2 m_K(\omega_{max} - \omega) \right\} \tag{129}$$

Finally, the energy correlation between pion and muon is

$$\rho(E,\omega) = \frac{4\pi^2 G'^2}{m_\mu m_\nu}\left\{4\left[2E\epsilon - m_K(\omega_{max}-\omega) - 4m_K m_\mu^2\epsilon + m_\mu^2 m_K(\omega_{max}-\omega)\right]\right.$$

$$\left.\left[1-\frac{\alpha}{\pi}\left\{2B(\beta)\log(m_K\delta E - m_\mu m_\nu)+C(\beta)+D(\beta)+\tfrac{1}{2}f_1\right\}\right]\right\} \tag{130}$$

Taking the ratio of $\rho(E,\omega)$ with $m_\nu \neq 0$ to that with $m_\nu = 0$, we have

$$\frac{\rho(E,\omega)_{m_\nu\neq 0}}{\rho(E,\omega)_{m_\nu=0}} = \frac{\left|T_o\right|^2_{m_\nu\neq 0}\left[1-\frac{\alpha}{\pi}\left\{2B(\beta)\log(m_K\delta E - m_\pi m_\nu)-C(\beta)-D(\beta)+\tfrac{1}{2}f_1\right\}\right]}{\left|T_o\right|^2_{m_\nu\neq 0}\left[1-\frac{\alpha}{\pi}\left\{2B(\beta)\log(m_K\delta E)-C(\beta)-D(\beta)+\tfrac{1}{2}f_1\right\}\right]} \tag{131}$$

We can see from eq. (129) that $\left|T_o\right|^2$ is a function of E and $\delta\omega = \omega_{max} - \omega$. The only place in this ratio where m_ν comes in is through the $\cos\eta$ integration and the Innerbremsstrahlung correction, i.e. $\log(m_K\delta E - m_\pi m_\nu)$. The value of $\cos\eta$ in the case $m_\nu \neq 0$ differs from the value in the case $m_\nu = 0$ by terms quadratic in m_ν and consequently we do not expect a large effect from the $\cos\eta$ term. The only important term appears to be the innerbremsstrahlung correction term $\log(m_K\delta E - m_\pi m_\nu)$. Recall that when $E = E_m - \delta E$, ω is very close to ω_m and we can now look at $\rho(E,\omega)$ as a function of E for a particular value of ω, say ω_1. Hence, the ratio of the energy correlations, for a particular value of pion energy, near the high muon energy end is

$$\frac{\rho(E_m,\omega_1)_{m_\nu\neq 0}}{\rho(E_m,\omega_1)_{m_\nu=0}} = 1-\frac{\alpha}{\pi}\left[2B(\beta_m)\log\left(1-\frac{m_\pi m_\nu}{m_K\delta E}\right)\right] \tag{132}$$

The reason for dropping the terms $C(\beta_m)$, $D(\beta_m)$ and f_1, and for taking the two $|T_o|^2$ to be equal is the same as that for the muon spectrum case (Section 5.1).

CHAPTER VI CONCLUSION AND DISCUSSION

The results of the calculations performed in the present work are displayed in eq. (119), eq. (122) and eq. (132) of Chapter V. We see from these equations that the radiative corrections to the ratio of the spectra with and without the neutretto mass comes in the form of the logarithmic term

$$\frac{\alpha}{\pi} B(\beta_m)\log(1-\frac{m_\pi m_\nu}{m_K \delta E}) \quad \text{or} \quad \frac{\alpha}{\pi} B(\beta_m)\log(1-\frac{m_\mu m_\nu}{m_K \delta\omega}).$$

This term was referred to as the universal logarithmic singularity in the paper by Allcock (1965)[3]. The presence of this term reflects the lack of cancellation of the infrared divergence. If we observe the spectrum of one of the decay particles, say the spectrum of the muon, near the high energy end, we expect the ratio $dP(E,m_\nu)/dP(E,0)$ to be deviated from that without corrections by an amount

$$\left[1-\frac{m_\pi^2 m_\nu^2}{m_K^2(\delta E)^2}\right]^{\frac{1}{2}}\left[-\frac{\alpha}{\pi} B(\beta_m)\log(1-\frac{m_\pi^2 m_\nu^2}{m_K^2(\delta E)^2})\right]$$

When $\delta E \rightarrow \frac{m_K}{m_\pi m_\nu}$, the logarithmic term tends to $-\infty$. But a measurement of the muon spectrum in the region where δE is very close to $m_K/m_\pi m_\nu$ would yield an upper limit of neutretto mass. The upper limit of m_ν

can be set by looking at the deviation from unity of the graph of the ratio $dP(E,m_\nu)/dP(E,0)$, plotted against the energy of the muon, near the end-point. The $dP(E,m_\nu)$ will be given directly from experiment, while $dP(E,0)$ can be calculated theoretically. Without the radiative corrections the ratio would be essentially the phase space ratio

$$\left[1 - \frac{m_\pi^2 m_\nu^2}{m_K^2(\delta E)^2}\right]^{\frac{1}{2}} ,$$

and that has been fully discussed in both the papers by Dennery and Primakoff (1963)[2] and by Ginsberg (1965)[5]. But if $B(\beta_m)$ is not too small, the contribution from the radiative corrections may become significant. In the $K_{\mu 3}$ decay, $m_K \gg m_\pi$ or m_μ, $\beta_m \approx 1$, thus $B(\beta_m)$ will be of the order of unity, so we expect the radiative corrections to be quite significant. For the muon spectrum, the correction to the ratio is 1% for $\delta E \approx 1.2 m_\pi m_\nu/m_K$ or $0.33\ m_\nu$. In the case of the pion spectrum, because of the presence of the term $m_\mu m_\nu$ as against the term $m_\pi m_\nu$ in the muon spectrum case we shall have to get closer to the end-point before we can see a significant departure from unity in the ratio. For instance, the correction is 1% for $\delta\omega \approx 1.2 m_\mu m_\nu/m_K$ or $0.25\ m_\nu$. Moreover the electrical neutrality of pion makes its spectrum experimentally less tractable than that of the muon. For the ratio of the energy correlations, the correction will be essentially the same as that in the muon spectrum case. However, the correlation experiments are more difficult to perform than the direct spectrum measurement.

BIBLIOGRAPHY

1. W.H. Barkas, W. Birnbaum and F.M. Smith, Phys. Rev., 101, 778 (1956).
2. P. Dennery and H. Primakoff, Phys. Lett., 6, 67 (1963).
3. G.R. Allcock, Proc. Phys. Soc., 85, 875 (1965).
4. A.N. Kamal, Nuovo Cimento, 40, 424 (1965).
5. E.S. Ginsberg, Nuovo Cimento, 36, 739 (1965).
6. CERN Lecture Note (1959).
7. S.S. Schweber, RELATIVISTIC QUANTUM FIELD THEORY, Chapter 15, Sect. C, p. 539, Harper and Row, Publishers, New York (1961).
8. R.P. Feynman, Phys. Rev., 76, 749 (1949).
9. J.M. Jauch and F. Rohrlich, THE THEORY OF PHOTON AND ELECTRON, Chapter 15, Sec. 2, p. 337, Addison-Wesley Publishing Company, Cambridge, Massachusetts (1955).
10. T. Kinoshita and A. Sirlin, Phys. Rev., 113, 1652 (1959), Appendix C.

APPENDIX A PHASE-SPACE INTEGRATION IN THE TIME-LIKE-G-FRAME

We have in Chapter I eq. (2) the phase-space integral for the muon spectrum.

$$dP(E,m_\nu) = \int \frac{d^3k}{2\omega}\,\frac{d^3q}{2\epsilon}\,\delta(G-k-q) \tag{A.1}$$

In which we have set G=K-p. Now, let us write $\frac{d^3q}{2\epsilon}$ as:

$$\frac{d^3q}{2} = d^3q\,dq^o\,\delta(q^{o^2} - q^2 - m_\nu^2) = d^3q\,dq^o\,\delta(q^{o^2} - \epsilon^2) \tag{A.2}$$

where $\epsilon^2=q^2+m^2$. Substituting eq. (A.2) into eq. (A.1), then we get

$$dP(E,m_\nu) = \int \frac{d^3k}{2\omega}\,d^4q\;\delta(q^{o2}-\epsilon^2)\,\delta^4(G-k-q) \tag{A.3}$$

In eq. (A.3) the q integration is trivial

$$dP(E,m_\nu) = \int \frac{d^3k}{2\omega}\,\delta\left[(G-k)^2 - m_\nu^2\right] \tag{A.4}$$

This expression is Lorentz invariant because both

$$\frac{d^3k}{2\omega} = d^4k\,\delta\left[k^{o2} - \omega^2\right] \quad \text{and} \quad \delta\left[(G-k)^2 - m_\nu^2\right]$$

[illegible]

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are Lorentz invariant.

So we can work in the time-like G frame i.e. $G_\mu = (G^o,0,0,0)$, where eq. (A.4) becomes

$$dP(E,m_\nu) = \int \frac{d^3k}{2}\,\delta\left[G_o^2 + k^2 - 2G^o k_o - m_\nu^2)\right]$$

$$= \frac{d^3k}{2\omega}\,\delta(G_o^2 - 2G^o k_o + m_\pi^2 - m_\nu^2)$$

$$= \frac{d^3k}{2\omega}\,\frac{1}{2G^o}\,\delta(k_o - \frac{G_o^2 + m_\pi^2 - m_\nu^2}{2G^o}) \qquad (A.5)$$

We can write $\bar{k}d\bar{k} = k^o dk^o$, so

$$\int \frac{d^3k}{2\omega} = 2\pi \int \bar{k}dk^o \qquad (A.6)$$

Substituting eq. (A.6) into eq. (A.5), we have

$$dP(E,m_\nu) = \int 2\pi\,\frac{\bar{k}dk^o}{2G^o}\,\delta(k_o - \frac{G_o^2+m_\pi^2-m_\nu^2}{2G^o})$$

$$= \pi\,\frac{k}{G^o}\Big|_{\text{evaluated at } k_o = \frac{G_o^2+m_\pi^2-m_\nu^2}{2G^o}} \qquad (A.7)$$

$$\bar{k}^2 = k_o^2 - m_\pi^2$$

$$= \left[\frac{G_o^2+m_\pi^2-m_\nu^2}{2G^o}\right]^2 - m_\pi^2$$

$$= \frac{1}{4G_o^2}\left[G_o^4 - 2G_o^2(m_\pi^2 + m_\nu^2) + (m_\pi^2 - m_\nu^2)^2\right] \tag{A.8}$$

Put this result into eq. (A.7) to obtain

$$dP(E,m_\nu) = \frac{\pi}{2G_o^2}\left[G_o^4 - 2G_o^2(m_\pi^2 + m_\nu^2) + (m_\pi^2 - m_\nu^2)^2\right]^{\frac{1}{2}}$$

This is in the time-like G frame, $G_o^2 = G^2$, so one can replace G_o^2 by G^2 all along.

$$dP(E,m_\nu) = \frac{\pi}{2G^2}\left[G^4 - 2G^2(m_\pi^2 + m_\nu^2) + (m_\pi^2 - m_\nu^2)^2\right]^{\frac{1}{2}} \tag{A.9}$$

APPENDIX B INTEGRATION OF $\int \left|\frac{p.\mathcal{E}^\lambda}{p.k'}\right|^2 \frac{d^3k'}{2\omega'}$ IN SPHERICAL COORDINATES

To evaluate this integral we assign to the photon a small mass λ so as to avert the divergence. Then $\omega'^2 = \bar{k}'^2 + \lambda^2$.

Let θ be the angle between $\bar{p}$ and $\bar{k}'$

$\bar{p}/E = \beta$ = velocity of the muon

$\bar{k}'/\omega' = \mho$, corresponds to the velocity of the photon.

Upon summing $\left|\frac{p.\mathcal{E}^\lambda}{p.k'}\right|^2$ over the polarization vectors of the photon, we have

$$\int \left|\frac{p.\mathcal{E}^\lambda}{p.k'}\right|^2 \frac{d^3k'}{2\omega'} = \tfrac{1}{2}\int \beta^2 \frac{(1-\mho^2\cos^2\theta)}{(1-\beta\mho\cos\theta)^2} \frac{d^3k'}{\omega'^3} \tag{B.1}$$

We take the direction of the $\bar{p}$ as the direction of the z axis, and do the integration in spherical coordinates.

$$\int \left|\frac{p.\mathcal{E}^\lambda}{p.k'}\right|^2 \frac{d^3k'}{2\omega'} = 2\pi \int_0^\pi \sin\theta d\theta \int_0^{\bar{K}'_{max}} \tfrac{1}{2}\beta^2 \frac{(1-\mho^2\cos^2\theta)}{(1-\beta\mho\cos\theta)^2} \frac{\bar{k}'^2 d\bar{k}'}{\omega'^3}$$

$$= 2\pi \int_0^\pi \sin\theta d\theta \int_0^{\bar{K}'_{max}} \tfrac{1}{2}\beta^2 \frac{(1-\mho^2\cos^2\theta)}{(1-\beta\mho\cos\theta)^2} \frac{{}^2 d\bar{k}'}{\omega'} \tag{B.2}$$

We can replace the t-integration by the $\mho$ integration in eq. (B.2)

Since $\upsilon = \bar{k}'/(\bar{k}'^2+\lambda^2)^{\frac{1}{2}}$ then $d\upsilon = \dfrac{\lambda^2}{(\bar{k}'^2+\lambda^2)^{3/2}}\, d\bar{k}'$

$$\frac{d\bar{k}'}{\omega'} = \frac{1}{1-\upsilon^2}\, d\upsilon$$

and $\upsilon=0$ when $\bar{k}'=0$, $\upsilon=1-\frac{1}{2}\bar{k}'_{max}$ when $\bar{k}'=\bar{k}'_{max}$.

Substituting these quantities into eq.(B.2) and letting $x=\cos\theta$ we obtain

$$\int \left|\frac{p.\varepsilon^\lambda}{p.k'}\right|^2 \frac{d^3k'}{2\omega'} = \pi \int_{-1}^{+1} dx \int_{0}^{1-\frac{1}{2}k'_{max}} \beta^2 \, \frac{(1-\upsilon^2x^2)\upsilon^2 d\upsilon}{(1-\beta\upsilon x)^2(1-\upsilon^2)}$$

$$= \int_{-1}^{+1} dx \int_{0}^{1-\frac{1}{2}k'_{max}} \frac{\pi\beta^2(1-x^2)\upsilon^2 d\upsilon}{(1-\beta\upsilon x)^2(1-\upsilon^2)} + \int_{-1}^{+1} dx \int_{0}^{1-\frac{1}{2}k'_{max}} \beta^2 \, \frac{\pi x^2\upsilon^2 d\upsilon}{(1-\beta\upsilon x)^2}$$

$$= I' + I'' \tag{B.3}$$

The integration of I' and I" are quite tedious, their results are given as follows:

$$I' = \pi\left\{2(\log 2 - 2)\left(-2-\frac{1}{\beta}\log\frac{1-\beta}{1+\beta}\right) - \tfrac{1}{2}\beta^2 \int_{-1}^{+1} \frac{1-x^2 dx}{(1-\beta x)^2} \log\frac{\lambda^2}{k'^2_{max}} \right. \tag{B.4}$$

$$\left. - \frac{1}{\beta}\left[L\left(\frac{1+\beta}{2}\right) - L\left(\frac{1-\beta}{2}\right)\right] + \frac{2}{\beta}\log\frac{1-\beta}{1+\beta} + \frac{1}{2\beta}\log\frac{1-\beta}{1+\beta}\log\frac{(1-\beta^2)}{4}\right\}$$

$$I'' = \pi\left\{2 - \frac{1}{\beta}\log\frac{1-\beta}{1+\beta} + \frac{2}{\beta}\left[L(\beta)-L(-\beta)\right]\right\} \tag{B.5}$$

where the L-functions are the Spence functions. Finally we have the expression

$$\int \left|\frac{p.\varepsilon^{\lambda}}{p.k'}\right|^2 \frac{d^3k'}{2\omega'} = \Big\{ \beta^2 \int_{-1}^{+1} \log \frac{\bar{k}'_{max}}{\lambda} \frac{1-x^2}{(1-\beta x)^2}\, dx - 4B(\beta)(\log 2-2) +$$

$$+ \frac{2}{\beta}\Big[L(\beta)-L(-\beta)\Big] - \frac{1}{\beta}\Big[L(\tfrac{1+\beta}{2})-L(\tfrac{1-\beta}{2})\Big] + 2 -$$

$$- \frac{1}{2\beta} \log \frac{1-\beta}{1+\beta} \left(\log \frac{1-\beta^2}{4} + 2\right)\Big\} \times \pi \tag{B.6}$$